Conformal Nets and Nocommutative Geometry

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Introduction

Conformal nets = description of (chiral) 2D CFT by means of algebras of bounded operators on Hilbert spaces (operator algebras: C*-algebras and von Neumann algebras).

Noncommutative geometry = study of operator algebras from a geometric point of view and of geometry from an operator algebraic point of view $\rightarrow$ noncommutative generalization of classical geometry. Here, in many cases, "noncommutative" should be understood in the weaker form "not necessarily commutative"

The theory of conformal nets is deeply related with various branches of the theory of operator algebras and in particular with subfactor theory.

Until recently, the possible relations between conformal nets and noncommutative geometry have not been investigated.

Aim of this talk: illustration of some of the main ideas underlying recent results in this direction following the "noncommutative geometrization program" for CFT through conformal nets an their representations (theory of superselection sectors) $\rightarrow$ connections between subfactor theory and noncommutative geometry.

This program was first proposed in Longo: CMP 2001, in agreement with previous suggestions by Doplicher (1985), and later developed in various directions in

- Longo, Kawahigashi: CMP 2005
- Carpi, Longo, Kawahigashi: AHP 2008
- Carpi, Hillier, Longo, Kawahigashi: CMP 2010
- Carpi, Conti, Hillier, Weiner: CMP 2013
- Carpi, Conti Hillier: AFA 2013
- Carpi, Hillier, Longo, Kawahigashi, Xu: arXiv:1207.2398
- Carpi, Hillier, Longo: arXiv:1304.4062
- Carpi, Weiner: In preparation

Classical mechanics

Motion of a classical particle on $\mathbb{R}$: p = momentum, q = position.

Classical phase space: $X = \{(p,q) : p \in \mathbb{R}, q \in \mathbb{R}\}$ = possible initial data for the Hamilton equations.

Observables: functions $F : X \to \mathbb{R}$. They form a commutative algebra over $\mathbb{R}$ (with obvious pointwise operations e.g. $FG(p,q) := F(p,q)G(p,q)$). Special cases: $(t = 0)$ momentum $P : (p,q) \mapsto p$ and position $Q : (p,q) \mapsto q$. Its complexification given by functions $F : X \to \mathbb{C}$ is a $*$-algebra with $*$-operation given by complex conjugation $F^*(p,q) = \overline{F(p,q)}$. Observables must be real functions $\leftrightarrow$ $F = F^*$.

States: Probability measures on X. The mean value of the observable F in the state μ is given by $\langle F\rangle_\mu = \int_X F \mathrm{d}\mu$.
medskip
Pure states: Dirac measures δ_x, where $x = (p,q)$, on X.
$\int_X F \mathrm{d}\delta_x = F(x)$.

Pure states $\leftrightarrow$ Optimal knowledge $\leftrightarrow$ Uncertainty free i.e. $\Delta_\mu F := \sqrt{\langle F^2\rangle_\mu - \langle F\rangle_\mu^2} = 0$ (standard deviation $= 0$) for all observables F if $\mu = \delta_x$ represents a pure state.

Quantum mechanics

Heisenberg uncertainty relation: For any physical state S the standard deviations $\Delta_S Q$, $\Delta_S P$ of the position and momentum observables obtained by the statistics on experimental data must satisfy

$$\Delta_S Q \Delta_S P \geq \frac{\hbar}{2}, \quad \text{where } \hbar = \frac{\text{Planck constant}}{2\pi} \sim 10^{-34} \quad \mathrm{J} \cdot \mathrm{s}$$

This is a fact of nature $\rightarrow$ new physics, new mathematical formalism.

Solution:

- Observable $\leftrightarrow$ selfadjoint operator $A = A^*$ on a complex Hilbert space $\mathcal{H}$
- Pure states $\leftrightarrow$ unit vector $\psi \in \mathcal{H}$
- Mean value $\leftrightarrow$ $\langle A \rangle_\psi = (\psi, A\psi)$
- Standard deviation $\leftrightarrow$ $\Delta_\psi A := \sqrt{\langle A^2 \rangle_\psi - \langle A \rangle_\psi^2}$.

A, B selfadjoint operators on $\mathcal{H}$ with commutator $[A, B] := AB - BA$, $\psi \in \mathcal{H}$, $\|\psi\| = 1$, then the Cauhy-Schwarz inequality $\rightarrow$

$$\Delta_\psi A \Delta_\psi B \geq \frac{1}{2} \langle [A, B] \rangle_\psi$$

Hence noncommutativity $\rightarrow$ uncertainty relations.
The Heisenberg uncertainty relation is satisfied for operators Q, P satisfying the canonical commutation relations

$[Q, P] = i\hbar \mathbf{1}$ $\leftrightarrow$ "Noncommutative phase space"

There is essentially a unique (up to unitary equivalence) "nice solution" (the Schrödinger representation):

$$\mathcal{H} = L^2(\mathbb{R}, dq), \ (Q\psi)(q) = q\psi(q), \ (P\psi)(q) = -i\hbar \frac{\mathrm{d}}{\mathrm{d}q} \psi(q)$$

Here Q, P are unbounded, densely defined selfadjoint operators.

Operator algebras

Message from quantum mechanics: $*$-algebras of operators $A : \mathcal{H} \to \mathcal{H}$ give a noncommutative analogues of the $*$-algebras of functions $F : X \to \mathbb{C}$. Real functions $F = F^*$ correspond to selfadjoint operators $A = A^*$.

Semplification: consider only bounded operators $A \in B(\mathcal{H})$ e.g. replace the observable Q by a $f(Q)$ with some increasing $f : \mathbb{R} \to (0, 1)$ (different labels of the experimental results).

$B(\mathcal{H})$ is an (associative) $*$-algebra with identity $\mathbf{1}$. It has various natural topologies, e.g. the norm topology and the strong operator topology.

A (selfadjoint) operator algebra is a $*$-subalgebra $\mathfrak{A} \subset B(\mathcal{H})$. $\mathfrak{A}$ is said to be unital if $\mathbf{1} \in \mathfrak{A}$.

- $\mathfrak{A}$ is a C*-algebra it is a closed subset of $B(\mathcal{H})$ with respect to the norm topology.
- $\mathfrak{A}$ is a von Neumann algebra if it is unital and it is a closed subset of $B(\mathcal{H})$ with respect to the strong operator topology.

We have defined C*-algebras and von Neumann algebras as represented in a given Hilbert space. In fact they can be characterized as abstract Banach algebras. In any case one can consider different Hilbert space representations $\pi : \mathfrak{A} \to B(\mathcal{H}_\pi)$ of a given operator algebra $\mathfrak{A}$.

Besides the quantum mechanics the formula operator algebras = noncommutative generalization of function algebras is strongly supported by the following fundamental result

Theorem (Gelfand-Naimark 1943)

Every commutative unital C-algebra $\mathfrak{A}$ is isometrically $*$-isomorphic to the algebra $C(X)$ of continuous complex valued functions on a compact Hausdorff space X (the spectrum of $\mathfrak{A}$). Every compact Hausdorff space X arises in this way. $\mathfrak{A}$ is separable if and only if X is metrizable. (For non unital $\mathfrak{A}$ there is a similar result with X locally compact)*

Accordingly: C*-algebras $\leftrightarrow$ Noncommutative topology

Although every von Neumann algebra is a unital C*-algebra the commutative von Neumann algebras (on a separable Hilbert space) are best described by the following theorem

Theorem

Every commutative von Neumann algebra $\mathfrak{A}$ on a separable Hilbert space $\mathcal{H}$ is isometrically $*$-isomorphic to the algebra $L^\infty(X, \mu)$ for some metrizable compact space X and some regular Borel probability measure μ on X. Every every pair (X, μ) with these properties arises in this way.

Accordingly: von Neumann algebras $\leftrightarrow$ Noncommutative measure (and probability) theory

K-theory

A central example of noncommutative topology is K-theory for C*-algebras (more generally for locally convex algebras).

If X is a compact Hausdorff space the equivalence classes of complex vector bundles over X generate an abelian group $K^0(X)$ through the operation $[E] + [F] = [E \oplus F]$.

If $\mathfrak{A}$ is a unital C*-algebra one can define an abelian group $K_0(\mathfrak{A})$ generated by suitable equivalence classes of projections $\mathrm{M}_\infty(\mathfrak{A})$ (the $*$-algebra of infinite matrices over $\mathfrak{A}$ with finitely many nonzero entries) and a natural operation $+$.

If $\mathfrak{A}$ is commutative and X is the spectrum of $\mathfrak{A}$ then $K_0(\mathfrak{A}) = K^0(X)$

Remark: one can consider K-theory also for algebras $\mathfrak{A}$ that are not C*-algebras but that are locally convex algebras, e.g. $\mathfrak{A} = C^\infty(X)$ with X smooth compact manifold.

K-theory plays a very important role in the theory operator algebras e.g. in the classification of C*-algebras and in noncommutative geometry.

Noncommutative geometry

Spectral triples: $(\mathfrak{A}, \mathcal{H}, D)$ also called K-cycles.

- $\mathfrak{A}$ unital $*$-algebra on $\mathcal{H}$
- D selfadjoint operator on $\mathcal{H}$ with compact resolvent, with domain $\mathrm{dom}(D) \subset \mathcal{H}$ (the Dirac operator) such that, $[A, D]$ is defined and bounded on $\mathrm{dom}(D)$.

The spectral triple is said to be even if there is selfadjoint operator Γ (grading operator) such that $\Gamma^2 = 1$, $\Gamma D \Gamma = -D$ and $[\Gamma, \mathfrak{A}] = \{0\}$.

The spectral triple is said to be θ-summable if $\mathrm{Tr}(e^{-\beta D^2}) < +\infty$ for all $\beta > 0$.

Remark: In general $\mathfrak{A}$ is not a C*-algebra.

Remark: It will be important to consider families of spectral triples over the same algebra $\mathfrak{A}$ by representing the latter in different Hilbert space. To emphasis this fact we will sometime use the notation $(\mathfrak{A}, (\pi, \mathcal{H}), D)$ for a spectral triple, where π is a representation of $\mathfrak{A}$ on $\mathcal{H}$.

Commutative example: Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ ($z \in S^1 \leftrightarrow z = e^{i\theta}$, $\theta \in \mathbb{R}$)

- $\mathcal{H} = L^2(S^1)$ (with normalized Lebesgue measure $\frac{\mathrm{d}\theta}{2\pi}$)
- $\mathfrak{A} = C^\infty(S^1)$ (acting on L^2 functions by pointwise multilication); it is a locally covex algebra
- $D = -i\frac{\mathrm{d}}{\mathrm{d}\theta}$

Theorem (Connes 2013)

Let $(\mathfrak{A}, \mathcal{H}, D)$ be a spectral triple with $\mathfrak{A}$ commutative + other conditions. Then $\mathfrak{A} = C^\infty(X)$ for some compact oriented smooth manifold X. Moreover, every compact oriented smooth manifold X appears in this way.

Accordingly: Spectral triples $\leftrightarrow$ Noncommutative differential geometry.

Entire cyclic cohomology

The entire cyclic cohomology of a locally convex algebra $\mathfrak{A}$ is a cohomology defined by certain sequences $\phi = (\phi_n)$ of multilinear forms on $\mathfrak{A}$ entire cochains).

- $CE^e(\mathfrak{A}) \equiv$ even entire cochains $\phi = (\phi_{2n})$
- $CE^o(\mathfrak{A}) \equiv$ odd entire cochains $\phi = (\phi_{2n+1})$
- $\partial : CE^e(\mathfrak{A}) \to CE^0(\mathfrak{A})$, $\partial : CE^o(\mathfrak{A}) \to CE^e(\mathfrak{A}) \equiv$ boundary operator
- $(HE^e(\mathfrak{A}), HE^o(\mathfrak{A})) \equiv$ entire cyclic cohomology $\equiv$ equivalence classes of cocycles $(\partial\phi = 0)$

Given an even cocycle $\phi \in CE^e(\mathfrak{A}) \cap \ker(\partial)$ and an idempotent $e \in \mathrm{M}_\infty(\mathfrak{A})$ one can define a complex number $\phi(e) \in \mathbb{C}$ which turns out to depend only on the cohomology class of ϕ in $HE^e(\mathfrak{A})$ and on the K-class of e in $K_0(\mathfrak{A})$ (pairing with K-theory).

JLO cocycle and index pairing

- JLO $\equiv$ Jaffe, Lesniewski, Osterwalder
- $\mathfrak{A} \equiv$ locally convex algebra
- $(\mathfrak{A}, (\pi, \mathcal{H}), D) \equiv$ even θ-summable spectral triple such that $A \mapsto \pi(A)$, $A \mapsto [D, \pi(A)]$ are continuous maps : $\mathfrak{A} \to B(\mathcal{H})$.
- $(\mathfrak{A}, (\pi, \mathcal{H}), D) \mapsto \tau$. $\tau \in CE^e(\mathfrak{A}) \cap \ker(\partial)$ even cocycle. (the JLO cocycle).
- $\tau(e) \in \mathbb{Z}$ for all idempotents in $\mathrm{M}_\infty(\mathfrak{A}) = \cup_{r \in \mathfrak{N}} \mathrm{M}_r(\mathfrak{A})$ (index pairing).
- $e \in \mathfrak{A}$ idempotent $\mathcal{H}_\pm := \ker(\Gamma \mp \mathbf{1})$ then

$$\tau(e) = \dim\ker\left((\pi(e)D\pi(e))|_{\pi(e)\mathcal{H}_+}\right) - \dim\ker\left((\pi(e)^* D\pi(e)^*)|_{\pi(e)\mathcal{H}_-}\right)$$

Fredholm index. A similar formula holds for idempotents in M_r, $r \in \mathbb{N}$.

Classical field theory

From now on $\hbar = 1$ and speed of light $= 1$.

- $\mathbb{M}_s \equiv s$-dimensional Minkowski spacetime $(s = 1 + d)$ with points $x = (t, \mathbf{x}) \in \mathbb{M}_s$
- $\Phi(x) \equiv$ field strenght at the spacetime point x. $\Phi(x)$ is an observable hence a functional : $X \to \mathbb{R}$, $X =$ phase space $= \{(\varphi, \pi)\}$ $=$ initial data. This means that $x \mapsto \Phi(x)[(\varphi, \pi)]$ is the solution of the wave equation with initial data $\Phi(0, \mathbf{x})[(\varphi, \pi)] = \varphi(\mathbf{x})$, $\partial_t \Phi(0, \mathbf{x})[(\varphi, \pi)] = \pi(\mathbf{x})$.
- Infinitely many degrees of freedom $\Rightarrow$ the phase space X is an infinite-dimensional manifold.
- The field $\Phi(x)$ at different spacetime points correspond to different observables which are measurable within different spacetime regions.
- Other observables are given by smeared fields $\Phi(f) := \int_{\mathbb{M}_s} \Phi(x) f(x) \mathrm{d}x$, $f \in C^\infty(\mathbb{M}_s)$.
- If $\mathrm{supp} f \subset \mathcal{O} \subset \mathbb{M}_s$ then $\Phi(f)$ is measurable within the spacetime region $\mathcal{O}$.
- Instead of $\mathbb{M}_s$ one can consider a curved spacetime $\mathcal{M}$.

Quantum field theory

- $x \mapsto \Phi(x)$ cannot be an operator valued function. In fact it gives an operator valued distribution (Wightman field) $f \mapsto \Phi(f)$, $f \in C_c^\infty(\mathbb{M}_s)$. Typically the smeared field $\Phi(f)$ is an unbounded operator.
- As in the case of finitely many degrees of freedom we can restrict to bounded functions of $\Phi(f)$.
- $\mathcal{A}(\mathcal{O}) \equiv$ von Neumann algebra generated by the bounded functions of the operators $\Phi(f)$ with $\mathrm{supp} f \subset \mathcal{O}$, $\mathcal{O} \subset \mathbb{M}_s$ open bounded spacetime region $\equiv$ algebra generated by observables measurable within $\mathcal{O}$.
- $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \equiv$ net of operator algebras (Haag-Kastler net) $\Rightarrow$ algebraic quantum field theory: Haag + Araki, Kastler, Schroer .....
- Locality: $[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = \{0\}$ if $\mathcal{O}_1$ is spacelike separated from $\mathcal{O}_2$.

Conformal nets on S^1

- Two-dimensional CFT $\equiv$ quantum field theories on $\mathbb{M}_2$ with scaling invariance $\Rightarrow$ certain relevant fields called (the chiral fields) depend only on $x - t$ (right-moving fields) or on $x + t$ (left-moving fields).
- Chiral CFT $\equiv$ CFT generated by left-moving (or right-moving) fields only. Chiral CFTs can be considered as QFTs on $\mathbb{R}$ and by conformal symmetry on its compactification S^1. Hence we can consider quantum fields on the unit circle $\Phi(z)$, $z \in S^1$ and the corresponding smeared fields $\Phi(f)$, $f \in C^\infty(S^1)$.
- The smeared fields $\Phi(f)$ generate conformal nets of von Neumann algebras on S^1 $\mathcal{A} : I \mapsto \mathcal{A}(I)$, $I \in \mathcal{I}$ ($\mathcal{I} \equiv$ family of open nonempty nondense intervals of S^1), acting on a separable Hilbert space $\mathcal{H}_\mathcal{A}$ (the vacuum Hilbert space).
- Conformal nets on on S^1 can be defined axiomatically.

Axioms of conformal nets on S^1

- **A.** Isotony. $I_1 \subset I_2 \Rightarrow \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$
- **B.** Locality. $I_1 \cap I_2 = \varnothing \Rightarrow [\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}$
- **C.** Conformal covariance. There exists a projective unitary rep. U of $\mathrm{Diff}(S^1)$ on $\mathcal{H}_{\mathcal{A}}$ such that

$$U(\gamma)\mathcal{A}(I)U(\gamma)^* \;=\; \mathcal{A}(\gamma I)$$

  and

$$(\gamma(z) = z \text{ for all } z \in S^1 \setminus I) \Rightarrow U(\gamma) \in \mathcal{A}(I).$$

- **D.** Positivity of the energy. U is a positive energy representation, i.e. the self-adjoint generator L_0 of the rotation subgroup of U (conformal Hamiltonian) has nonnegative spectrum.
- **E.** Vacuum. $\mathrm{Ker}(L_0) = \mathbb{C}\Omega$, where Ω (the vacuum vector) is a unit vector in $\mathcal{H}_{\mathcal{A}}$.
- **F.** Irreducibility. $\mathcal{H}_{\mathcal{A}}$ and $\{0\}$ are the only closed subspaces invariant for the action of all the algebras $\mathcal{A}(I)$.

Virasoro algebra

- The projective unitary representation U of $\mathrm{Diff}(S^1)$ gives rise to a representation on (a dense subspace) of $\mathcal{H}_{\mathcal{A}}$ of the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{-n,m}\mathbf{1} \quad n, m \in \mathbb{Z}$$

  with central charge c.
- The Virasoro field $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ is the chiral energy-momentum tensor of the theory.
- If the representation of the Virasoro algebra on $\mathcal{H}_{\mathcal{A}}$ is (topologically) irreducible then the net $\mathcal{A}$ is generated by the field $L(z)$ and is called the Virasoro net with central charge c

Representations of conformal nets

- A representation of a conformal net $\mathcal{A}$ on S^1 is a family $\pi = \{\pi_I : I \in \mathcal{I}\}$ of representations π_I of $\mathcal{A}(I)$ on a common Hilbert space $\mathcal{H}_\pi$ such that $I_1 \subset I_2 \Rightarrow \pi_{I_2|\mathcal{A}(I_1)} = \pi_{I_1}$.
- The equivalence class $[\pi]$ of an irreducible representation on a separable $\mathcal{H}_\pi$ is called a sector of the conformal net $\mathcal{A}$.
- The identical representation π_0 of $\mathcal{A}$ on the vacuum Hilbert space $\mathcal{H}_\mathcal{A}$ is called the vacuum representation and the corresponding sector $[\pi_0]$ the vacuum sector.
- π is said to be localized in a given interval I_0 if $\mathcal{H}_\pi = \mathcal{H}_\mathcal{A}$ and $\pi_{I_1}(x) = x$ whenever $I_1 \cap I_0 = \varnothing$ and $x \in \mathcal{A}(I_1)$. Then it follows that $\pi_I(\mathcal{A}(I)) \subset \mathcal{A}(I)$ for all I containing I_0, namely π_I is an endomorphism of $\mathcal{A}(I)$ for all $I \supset I_0$.

Universal algebras and DHR endomorphisms

The universal C*-algebra of $\mathcal{A}$ (Fredenhagen) can be defined as the unique (up to isomorphism) unital C*-algebra $C^*(\mathcal{A})$ such that

- there are unital embeddings $\iota_I : \mathcal{A}(I) \to C^*(\mathcal{A}),\ I \in \mathcal{I}$ such that $\iota_{I_2|\mathcal{A}(I_1)} = \iota_{I_1}$ if $I_1 \subset I_2$, and all $\iota_I(\mathcal{A}(I)) \subset C^*(\mathcal{A})$ together generate $C^*(\mathcal{A})$ as a C*-algebra;
- for every representation π of $\mathcal{A}$ on $\mathcal{H}_\pi$, there is a unique strongly continuous representation (denoted by the same symbol) $\pi : C^*(\mathcal{A}) \to B(\mathcal{H}_\pi)$ such that $\pi_I = \pi \circ \iota_I, \quad I \in \mathcal{I}$.

The universal von Neumann algebra of $\mathcal{A}$ is the so called enveloping von Neumann algebra of $C^*(A)$. It can be defined as the unique (up to isomorphism) unital von Neumann algebra $W^*(\mathcal{A})$ such that

- $C^*(\mathcal{A})$ embeds in $W^*(\mathcal{A})$ as a strongly dense subalgebra.
- For every representation π of $C^*(\mathcal{A})$ on $\mathcal{H}_\pi$, there is a unique representation (denoted by the same symbol) $\pi : W^*(\mathcal{A}) \to B(\mathcal{H}_\pi)$ extending π and which is continuous when $W^*(\mathcal{A})$ and $B(\mathcal{H}_\pi)$ are endowed with the strong operator topology.

We will sometime indentify $\iota_I(\mathcal{A}(I))$ with $\mathcal{A}(I)$ so that the latter will be considered as a subalgebra of $C^*(\mathcal{A})$ and hence of $W^*(\mathcal{A})$.
There is a canonical correspondence between localized representations of a conformal net $\mathcal{A}$ and DHR (localized and transportable) endomorphisms of $C^*(\mathcal{A})$. If π is a representation of $\mathcal{A}$ localized in $I \in \mathcal{I}$ then the corresponding DHR endomorphism ρ satisfies

$$\pi = \pi_0 \circ \rho$$

The DHR endomorphism corresponding to π_0 is the identical endomorphism ι.

DHR $\equiv$ Doplicher, Haag and Roberts.

The geometrization program

- Consider the algebras associated to a conformal net $\mathcal{A}$ $(\mathcal{A}(I),\ I \in \mathcal{I};\ C^*(\mathcal{A});\ W^*(\mathcal{A})\dots)$ as algebras of functions on an infinite-dimensional manifold (the phase space of the theory)
- Use the representation theory of $\mathcal{A}$ to define θ-summable spectral triples on appropriate smooth/differentiable subalgebras.
- Consider the corresponding JLO cocycle as noncommutative geometric invariants associated to interesting families of representations and find examples where one can prove, using the index pairing with K-theory, that different representations of the net give rise to different entire cohomolgy classes.
- Strategy: use supersymmetric extensions of the conformal symmetry: $N = 1$ or $N = 2$ super-Virasoro algebras
- Related noncommutative topological investigations: study the action of DHR endomorphism on suitable algebras associated with $\mathcal{A}$ and investigate on the possibility to get interesting actions on the corresponding K-theory. Analyze possible KK-theoretical interpretations.

Few more details

Let $\mathcal{A}$ be a conformal net admitting, in an appropriate sense, a supersymmetric extension of the conformal symmetry. Then, in various cases one can prove that the net admit special representations π (that I will call here minimally reducible Ramond type representations) with the following properties:

- $\mathcal{H}_\pi$ is graded by a selfadjoint unitary Γ_π commuting with $\pi(W^*(\mathcal{A}))$
- There is a selfadjoint operator Q_π (the supercharge operator) anti-commuting with Γ_π and such that $Q_\pi^2 = L_0^\pi - \frac{c}{24}\mathbf{1}$, where L_0^π is the conformal Hamiltonian for the (covariant) representation π.
- $\mathrm{Tr}(e^{-\beta L_0^\pi}) < +\infty$ for all $\beta > 0$.
- The subrepresentations $\pi_\pm$ of $W^*(\mathcal{A})$ on $\mathcal{H}_{\pi,\pm} := \ker(\Gamma_\pi \mp \mathbf{1})$ are irreducible and mutually inequivalent.

Remark. These representations arise from soliton representations of a graded-local superconformal extension $\mathcal{F} \supset \mathcal{A}$.

We considered two strategies.

Strategy **1**.

- $\Delta_R \equiv$ family mutually inequivalent minimally reducible Ramond type representations.
- $\mathfrak{A}_{\Delta_R} \equiv \{A \in W^*(\mathcal{A}) : [Q_\pi, \pi(A)]$ bounded on $\mathrm{dom}(Q_\pi)$ $\forall \pi \in \Delta_R\}$
- Natural locally convex topology on $\mathfrak{A}_{\Delta_R}$.
- $(\mathfrak{A}_{\Delta_R}, (\pi, \mathcal{H}_\pi), Q_\pi))$ θ-summable even spectral triple with the right continuity properties for all $\pi \in \Delta_R \Rightarrow$ JLO cocycle τ_π for all $\pi \in \Delta_R$.
- The cohomology classes of the cocycles τ_π are separated by suitable projections in $\mathfrak{A}_{\Delta_R}$.

This strategy has been undertaken successfully in Carpi, Hillier, Longo, Kawahigashi, Xu: arXiv:1207.2398 in the case of (the Bose part of) $N = 2$ super-Virasoro nets. In particular, for these models, the algebras $\mathfrak{A}_{\Delta_R}$ have nontrivial K_0 group.

Strategy **2**.

- Consider a fixed minimally reducible Ramond type representations π and consider a family Δ of DHR endomorphisms of $C^*(\mathcal{A})$, possibly satisfying suitable "differentiability" conditions .
- $\mathfrak{A}_\Delta \equiv \{A \in W^*(\mathcal{A}) : [Q_\pi, \pi \circ \rho(A)]$ bounded on $\mathrm{dom}(Q_\pi)\ \forall \rho \in \Delta\}$
- Natural locally convex topology on $\mathfrak{A}_\Delta$.
- $(\mathfrak{A}_\Delta, (\pi \circ \rho, \mathcal{H}_\pi), Q_\pi))$ θ-summable even spectral triple with the right continuity properties for all $\rho \in \Delta \Rightarrow$ JLO cocycle τ_ρ for all $\rho \in \Delta$.
- The cohomology classes of the cocycles τ_ρ are separated by suitable projections in $\mathfrak{A}_\Delta$.

This strategy has been undertaken successfully in Carpi, Hillier, Longo: arXiv:1304.4062 in the case of (the Bose part of) $N = 1$ super-Virasoro nets and supersymmetric loop group models. In particular, for these models, the algebras $\mathfrak{A}_\Delta$ have nontrivial K_0 group.

THANK YOU VERY MUCH!