Noncommutative Geometry and Physics

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Variables

One striking point is the role that "variables" play in Newton's approach, while Leibniz introduced the term "infinitesimal" but did not use variables. According to Newton :

"In a certain problem, a variable is the quantity that takes an infinite number of values which are quite determined by this problem and are arranged in a definite order"

"A variable is called infinitesimal if among its particular values one can be found such that this value itself and all following it are smaller in absolute value than an arbitrary given number"

Classical formulation

In the classical formulation of variables as maps from a set X to the real numbers $\mathbb{R}$, the set X has to be uncountable if some variable has continuous range. But then for any other variable with countable range some of the multiplicities are infinite. This means that discrete and continuous variables cannot coexist in this modern formalism. Fortunately everything is fine and this problem of treating continuous and discrete variables on the same footing is completely solved using the formalism of quantum mechanics.

Quantum formalism

The first basic change of paradigm has indeed to do with the classical notion of a “real variable” which one would classically describe as a real valued function on a set X, ie as a map from this set X to real numbers. In fact quantum mechanics provides a very convenient substitute. It is given by a self-adjoint operator in Hilbert space. Note that the choice of Hilbert space is irrelevant here since all separable infinite dimensional Hilbert spaces are isomorphic.

All the usual attributes of real variables such as their range, the number of times a real number is reached as a value of the variable etc... have a perfect analogue in the quantum mechanical setting. The range is the spectrum of the operator, and the spectral multiplicity gives the number of times a real number is reached. In the early times of quantum mechanics, physicists had a clear intuition of this analogy between operators in Hilbert space (which they called q-numbers) and variables.

Infinitesimal variables

What is surprising is that the new set-up immediately provides a natural home for the "infinitesimal variables" and here the distinction between "variables" and numbers (in many ways this is where the point of view of Newton is more efficient than that of Leibniz) is essential.

Indeed it is perfectly possible for an operator to be “smaller than epsilon for any epsilon” without being zero. This happens when the norm of the restriction of the operator to subspaces of finite codimension tends to zero when these subspaces decrease (under the natural filtration by inclusion). The corresponding operators are called “compact” and they share with naive infinitesimals all the expected algebraic properties. Indeed they form a two-sided ideal of the algebra of bounded operators in Hilbert space and the only property of the naive infinitesimal calculus that needs to be dropped is the commutativity.

Discrete and continuous coexist

It is only because one drops commutativity that variables with continuous range can coexist with variables with countable range.

Thus it is the uniqueness of the separable infinite dimensional Hilbert space that cures the above problem, $L^2[0,1]$ is the same as $\ell^2(\mathbb{N})$, and variables with continuous range coexist happily with variables with countable range, such as the infinitesimal ones. The only new fact is that they do not commute, and the real subtlety is in their algebraic relations. For instance it is the lack of commutation of the line element ds with the coordinates that allows one to measure distances in a noncommutative space given as a spectral triple.

Space X	**Algebra** $\mathcal{A}$
Real variable x^μ	**Self-adjoint** T
Set of values	**Spectrum of** T
Infinitesimal	**Compact** ϵ
Order α	$\mu_n(\epsilon) = O(n^{-\alpha})$
Integral of infinitesimal	$⨍ \epsilon =$ **Coefficient of** $\log(\Lambda)$ **in** $\mathrm{Tr}_\Lambda(\epsilon)$
Line element $\sqrt{g_{\mu\nu}\, dx^\mu dx^\nu}$	**ds** $=$ **Fermion propagator**

Variability

At the philosophical level there is something quite satisfactory in the variability of the quantum mechanical observables. Usually when pressed to explain what is the cause of the variability in the external world, the answer that comes naturally to the mind is just : the passing of time.

But precisely the quantum world provides a more subtle answer since the reduction of the wave packet which happens in any quantum measurement is nothing else but the replacement of a "q-number" by an actual number which is chosen among the elements in its spectrum. Thus there is an intrinsic variability in the quantum world which is so far not reducible to anything classical. The results of observations are intrinsically variable quantities, and this to the point that their values cannot be reproduced from one experiment to the next, but which, when taken altogether, form a q-number.

How can time emerge ?

Quantum thermodynamics, Ludwig Boltzmann

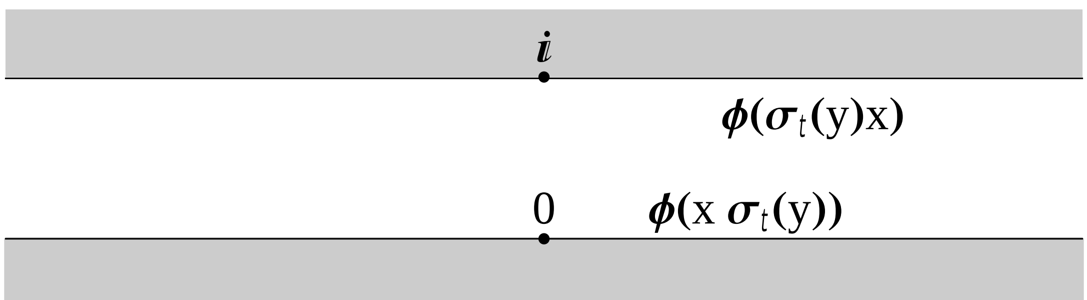
$$\varphi(A) = Z^{-1}\,\mathrm{tr}(A\,e^{-\beta H})$$

$$Z = \mathrm{tr}(e^{-\beta H})$$

The KMS condition

$$\varphi(x^*x) \geq 0 \qquad \forall\, x \in \mathcal{A}\,,\ \varphi(1) = 1\,.$$

$$\sigma_t \in \mathrm{Aut}(\mathcal{A})$$



$$F_{x,y}(t) = \varphi(x\sigma_t(y))$$

$$F_{x,y}(t + i\beta) = \varphi(\sigma_t(y)x), \qquad \forall t \in \mathbb{R}.$$

Tomita-Takesaki (1967)

Theorem

Let M be a von Neumann algebra and φ a faithful normal state on M, then there exists a unique one parameter group

$$\sigma_t^{\varphi} \in \mathrm{Aut}(M)$$

which fulfills the KMS condition for $\beta = 1$.

Thesis (1972)

Theorem (alain connes)

$$1 \to \text{Int}(M) \to \text{Aut}(\mathcal{M}) \to \text{Out}(M) \to 1,$$

The class of σ_t^{φ} in $\text{Out}(M)$ does not depend upon the choice of the state φ.

Thus a von Neumann algebra M, possesses a canonical time evolution

$$\mathbb{R} \xrightarrow{\delta} \text{Out}(M).$$

Noncommutativity $\Rightarrow$ Time Evolution

Many mathematical corollaries but what about physics ?

1. We (with Carlo Rovelli) interpret time as a one parameter group of automorphisms of the algebra of observables for gravitation.
2. Thermodynamical origin.

Algebra of observables in QG ?

Find complete invariants of geometric spaces

How can we invariantly specify a point in a geometric space ?

It is well known since a famous one page paper of John Milnor that the spectrum of operators, such as the Laplacian, does not suffice to characterize a compact Riemannian space. But it turns out that the missing information is encoded by the relative position of two abelian algebras of operators in Hilbert space. Due to a theorem of von Neumann the algebra of multiplication by all measurable bounded functions acts in Hilbert space in a unique manner, independent of the geometry one starts with. Its relative position with respect to the other abelian algebra given by all functions of the Laplacian suffices to recover the full geometry, provided one knows the spectrum of the Laplacian. For some reason which has to do with the inverse problem, it is better to work with the Dirac operator.

The unitary (CKM) invariant of Riemannian manifolds

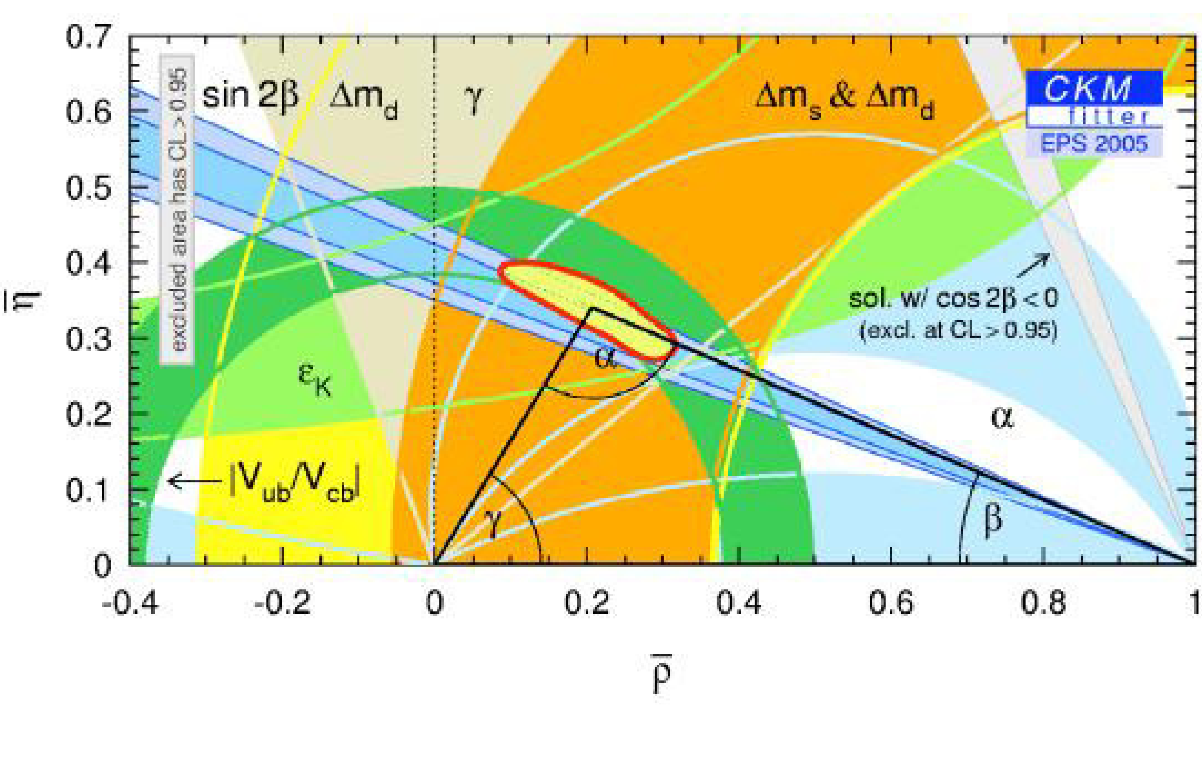
The invariants are :

– The spectrum Spec(D).

– The relative spectrum Spec$_N(M)$ $(N = \{f(D)\})$.

Flavor changing weak decays

$$\frac{ig}{2\sqrt{2}}W_\mu^+\left(\bar{u}_j^\lambda\gamma^\mu(1+\gamma^5)C_{\lambda\kappa}d_j^\kappa\right)+$$
$$\frac{ig}{2\sqrt{2}}W_\mu^-\left(\bar{d}_j^\kappa C_{\kappa\lambda}^\dagger\gamma^\mu(1+\gamma^5)u_j^\lambda\right)$$



Cabibbo-Kobayashi-Maskawa

$$C = \begin{bmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{bmatrix}$$

$$C = \begin{bmatrix} C_{ud} & C_{us} & C_{ub} \\ C_{cd} & C_{cs} & C_{cb} \\ C_{td} & C_{ts} & C_{tb} \end{bmatrix}$$

$$C = \begin{bmatrix} c_1 & -s_1 c_3 & -s_1 s_3 \\ s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e_\delta & c_1 c_2 s_3 + s_2 c_3 e_\delta \\ s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e_\delta & c_1 s_2 s_3 - c_2 c_3 e_\delta \end{bmatrix}$$

$c_i = \cos\theta_i$, $s_i = \sin\theta_i$, and $e_\delta = \exp(i\delta)$

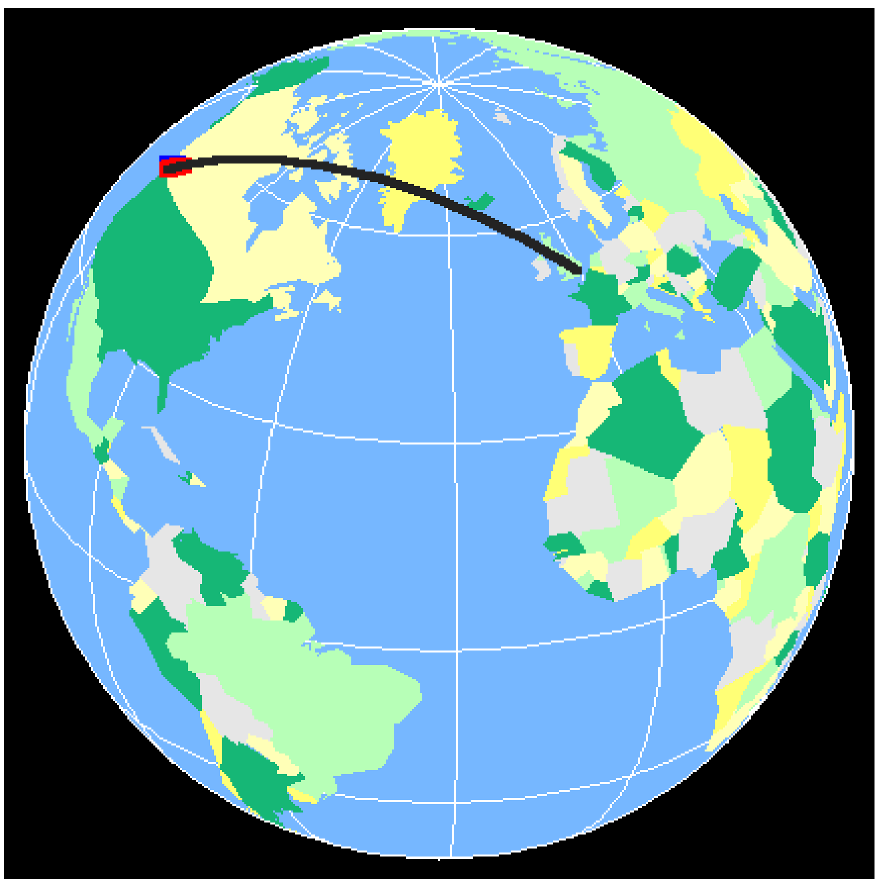
Points

Once we know the spectrum Λ of D, the missing information is contained in $\mathrm{Spec}_N(M)$.

It should be interpreted as giving the probability for correlations between the possible frequencies, while a "point" of the geometric space X can be thought of as a correlation, *i.e.* a specific positive hermitian matrix $\rho_{\lambda\kappa}$ (up to scale) in the support of ν.

What is a metric in spectral geometry

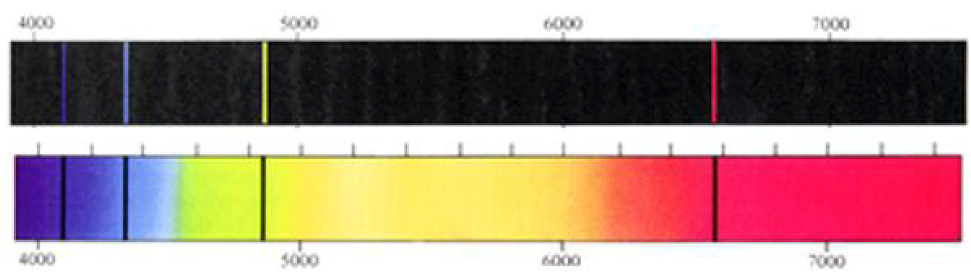
$$d(A,B) = \text{Inf} \int_\gamma \sqrt{g_{\mu\,\nu}\, dx^\mu\, dx^\nu}$$



Dirac's square root of the Laplacian

$$(\mathcal{A}, \mathcal{H}, D)\,, \qquad ds = D^{-1}\,,$$

$$d(A,B) = \text{Sup}\,\{|f(A) - f(B)|\,;\; f \in \mathcal{A}\,,\; \|[D,f]\| \leq 1\,\}$$



Meter $\rightarrow$ Wave length (Krypton (1967) spectrum of 86Kr then Caesium (1984) hyperfine levels of C133)

Space-Time

Joint work with **Ali Chamseddine**

Our knowledge of spacetime is described by two existing theories :
– General Relativity
– The Standard Model of particle physics

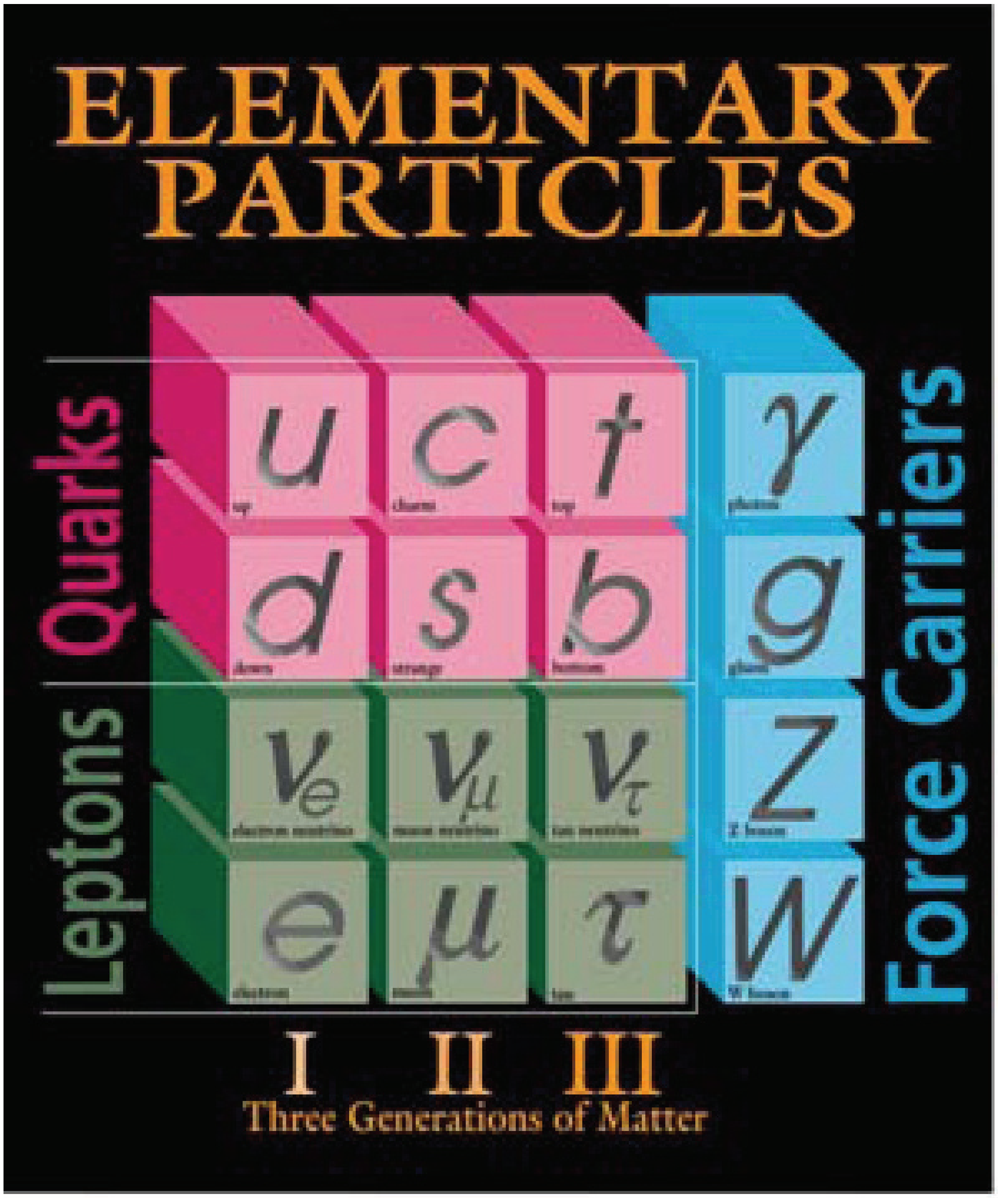
Curved Space, gravitational potential $g_{\mu\nu}$

$$ds^2 = g_{\mu\nu} dx^\mu \, dx^\nu$$

Action principle

$$S_E[\, g_{\mu\nu}] = \frac{1}{G} \int_M r \, \sqrt{g} \; d^4x$$

$$S = S_E + S_{SM}$$



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Standard Model

$$\mathcal{L}_{SM} = -\frac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \frac{1}{4}g_s^2 f^{abc} f^{ade} g^b_\mu g^c_\nu g^d_\mu g^e_\nu$$

$$-\partial_\nu W^+_\mu \partial_\nu W^-_\mu - M^2 W^+_\mu W^-_\mu - \frac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \frac{1}{2c_w^2} M^2 Z^0_\mu Z^0_\mu - \frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu$$

$$-igc_w(\partial_\nu Z^0_\mu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu (W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu)$$

$$+Z^0_\mu (W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - igs_w(\partial_\nu A_\mu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu)$$

$$-A_\nu (W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + A_\mu (W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu))$$

$$-\frac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \frac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu$$

$$+g^2 c_w^2 (Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s_w^2 (A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu)$$

$$+g^2 s_w c_w (A_\mu Z^0_\nu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu) - \frac{1}{2}\partial_\mu H \partial_\mu H - \frac{1}{2}m_h^2 H^2$$

$$-\partial_\mu \phi^+ \partial_\mu \phi^- - M^2 \phi^+ \phi^- - \frac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 - \frac{1}{2c_w^2} M^2 \phi^0 \phi^0$$

$$-\beta_h \left(\frac{2M^2}{g^2} + \frac{2M}{g}H + \frac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-) \right) + \frac{2M^4}{g^2}\alpha_h$$

$$-g\alpha_h M\left(H^3+H\phi^0\phi^0+2H\phi^+\phi^-\right)$$

$$-\frac{1}{8}g^2\alpha_h\left(H^4+(\phi^0)^4+4(\phi^+\phi^-)^2+4(\phi^0)^2\phi^+\phi^-+4H^2\phi^+\phi^-+2(\phi^0)^2H^2\right)$$

$$-gMW_\mu^+W_\mu^-H-\frac{1}{2}g\frac{M}{c_w^2}Z_\mu^0Z_\mu^0H$$

$$-\frac{1}{2}ig\left(W_\mu^+(\phi^0\partial_\mu\phi^--\phi^-\partial_\mu\phi^0)-W_\mu^-(\phi^0\partial_\mu\phi^+-\phi^+\partial_\mu\phi^0)\right)$$

$$+\frac{1}{2}g\left(W_\mu^+(H\partial_\mu\phi^--\phi^-\partial_\mu H)+W_\mu^-(H\partial_\mu\phi^+-\phi^+\partial_\mu H)\right)$$

$$+\frac{1}{2}g\frac{1}{c_w}(Z_\mu^0(H\partial_\mu\phi^0-\phi^0\partial_\mu H)-ig\frac{s_w^2}{c_w}MZ_\mu^0(W_\mu^+\phi^--W_\mu^-\phi^+)$$

$$+igs_wMA_\mu(W_\mu^+\phi^--W_\mu^-\phi^+)-ig\frac{1-2c_w^2}{2c_w}Z_\mu^0(\phi^+\partial_\mu\phi^--\phi^-\partial_\mu\phi^+)$$

$$+igs_wA_\mu(\phi^+\partial_\mu\phi^--\phi^-\partial_\mu\phi^+)-\frac{1}{4}g^2W_\mu^+W_\mu^-\left(H^2+(\phi^0)^2+2\phi^+\phi^-\right)$$

$$-\frac{1}{8}g^2\frac{1}{c_w^2}Z_\mu^0Z_\mu^0\left(H^2+(\phi^0)^2+2(2s_w^2-1)^2\phi^+\phi^-\right)$$

$$-\frac{1}{2}g^2\frac{s_w^2}{c_w}Z_\mu^0\phi^0(W_\mu^+\phi^-+W_\mu^-\phi^+)-\frac{1}{2}ig^2\frac{s_w^2}{c_w}Z_\mu^0H(W_\mu^+\phi^--W_\mu^-\phi^+)$$

$$+\frac{1}{2}g^2s_wA_\mu\phi^0(W_\mu^+\phi^-+W_\mu^-\phi^+)+\frac{1}{2}ig^2s_wA_\mu H(W_\mu^+\phi^--W_\mu^-\phi^+)$$

$$-g^2\frac{s_w}{c_w}(2c_w^2-1)Z_\mu^0A_\mu\phi^+\phi^--g^2s_w^2A_\mu A_\mu\phi^+\phi^-$$

$$+\frac{1}{2}ig_s\lambda^a_{ij}(\bar{q}^\sigma_i\gamma^\mu q^\sigma_j)g^a_\mu - \bar{e}^\lambda(\gamma\partial + m^\lambda_e)e^\lambda - \bar{\nu}^\lambda\gamma\partial\nu^\lambda - \bar{u}^\lambda_j(\gamma\partial + m^\lambda_u)u^\lambda_j$$

$$-\bar{d}^\lambda_j(\gamma\partial + m^\lambda_d)d^\lambda_j + igs_wA_\mu\left(-(\bar{e}^\lambda\gamma^\mu e^\lambda) + \frac{2}{3}(\bar{u}^\lambda_j\gamma^\mu u^\lambda_j) - \frac{1}{3}(\bar{d}^\lambda_j\gamma^\mu d^\lambda_j)\right)$$

$$+\frac{ig}{4c_w}Z^0_\mu\{(\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{e}^\lambda\gamma^\mu(4s^2_w - 1 - \gamma^5)e^\lambda)$$

$$+(\bar{d}^\lambda_j\gamma^\mu(\frac{4}{3}s^2_w - 1 - \gamma^5)d^\lambda_j) + (\bar{u}^\lambda_j\gamma^\mu(1 - \frac{8}{3}s^2_w + \gamma^5)u^\lambda_j)\}$$

$$+\frac{ig}{2\sqrt{2}}W^+_\mu\left((\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)e^\lambda) + (\bar{u}^\lambda_j\gamma^\mu(1+\gamma^5)C_{\lambda\kappa}d^\kappa_j)\right)$$

$$+\frac{ig}{2\sqrt{2}}W^-_\mu\left((\bar{e}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{d}^\kappa_jC^\dagger_{\kappa\lambda}\gamma^\mu(1+\gamma^5)u^\lambda_j)\right)$$

$$+\frac{ig}{2\sqrt{2}}\frac{m^\lambda_e}{M}\left(-\phi^+(\bar{\nu}^\lambda(1-\gamma^5)e^\lambda) + \phi^-(\bar{e}^\lambda(1+\gamma^5)\nu^\lambda)\right)$$

$$-\frac{g}{2}\frac{m^\lambda_e}{M}\left(H(\bar{e}^\lambda e^\lambda) + i\phi^0(\bar{e}^\lambda\gamma^5 e^\lambda)\right)$$

$$+\frac{ig}{2M\sqrt{2}}\phi^+\left(-m^\kappa_d(\bar{u}^\lambda_jC_{\lambda\kappa}(1-\gamma^5)d^\kappa_j) + m^\lambda_u(\bar{u}^\lambda_jC_{\lambda\kappa}(1+\gamma^5)d^\kappa_j\right)$$

$$+\frac{ig}{2M\sqrt{2}}\phi^-\left(m^\lambda_d(\bar{d}^\lambda_jC^\dagger_{\lambda\kappa}(1+\gamma^5)u^\kappa_j) - m^\kappa_u(\bar{d}^\lambda_jC^\dagger_{\lambda\kappa}(1-\gamma^5)u^\kappa_j\right)$$

$$-\frac{g}{2}\frac{m^\lambda_u}{M}H(\bar{u}^\lambda_ju^\lambda_j) - \frac{g}{2}\frac{m^\lambda_d}{M}H(\bar{d}^\lambda_jd^\lambda_j) + \frac{ig}{2}\frac{m^\lambda_u}{M}\phi^0(\bar{u}^\lambda_j\gamma^5u^\lambda_j) - \frac{ig}{2}\frac{m^\lambda_d}{M}\phi^0(\bar{d}^\lambda_j\gamma^5d^\lambda_j)$$

Let us consider the simplest example

$$\mathcal{A} = C^\infty(M, M_n(\mathbb{C})) = C^\infty(M) \otimes M_n(\mathbb{C})$$

Algebra of $n \times n$ matrices of smooth functions on manifold M.

The group $\mathrm{Int}(\mathcal{A})$ of inner automorphisms is locally isomorphic to the group $\mathcal{G}$ of smooth maps from M to the small gauge group $SU(n)$

$$1 \to \mathrm{Int}(\mathcal{A}) \to \mathrm{Aut}(\mathcal{A}) \to \mathrm{Out}(\mathcal{A}) \to 1$$

becomes identical to

$$1 \to \mathrm{Map}(M, G) \to \mathcal{G} \to \mathrm{Diff}(M) \to 1.$$

We have shown that the study of pure gravity on this space yields Einstein gravity on M minimally coupled with Yang-Mills theory for the gauge group $SU(n)$. The Yang-Mills gauge potential appears as the inner part of the metric, in the same way as the group of gauge transformations (for the gauge group $SU(n)$) appears as the group of inner diffeomorphisms.

The restriction to spin manifolds is obtained by requiring a *real structure i.e.* an antilinear unitary operator J acting in $\mathcal{H}$ which plays the same role and has the same algebraic properties as the charge conjugation operator in physics.

The following further relations hold for D, J and γ

$$J^2 = \varepsilon\,,\; DJ = \varepsilon' JD, \quad J\,\gamma = \varepsilon'' \gamma J, \quad D\gamma = -\gamma D$$

The values of the three signs $\varepsilon, \varepsilon', \varepsilon''$ depend only, in the classical case of spin manifolds, upon the value of the dimension n modulo 8 and are given in the following table :

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

In the classical case of spin manifolds there is thus a relation between the metric (or spectral) dimension given by the rate of growth of the spectrum of D and the integer modulo 8 which appears in the above table. For more general spaces however the two notions of dimension (the dimension modulo 8 is called the KO-dimension because of its origin in K-theory) become independent since there are spaces F of metric dimension 0 but of arbitrary KO-dimension.

Starting with an ordinary spin geometry M of dimension n and taking the product $M \times F$, one obtains a space whose metric dimension is still n but whose KO-dimension is the sum of n with the KO-dimension of F.

As it turns out the Standard Model with neutrino mixing favors the shift of dimension from the 4 of our familiar space-time picture to $10 = 4 + 6 = 2$ modulo 8.

The shift from 4 to 10 is a recurrent idea in string theory compactifications, where the 6 is the dimension of the Calabi-Yau manifold used to “compactify”. The difference of this approach with ours is that, in the string compactifications, the metric dimension of the full space-time is now 10 which can only be reconciled with what we experience by requiring that the Calabi-Yau fiber remains unnaturally small.

In order to learn how to perform the above shift of dimension using a 0-dimensional space F, it is important to classify such spaces. This was done in joint work with A. Chamseddine. We classified there the *finite* spaces F of given KO-dimension. A space F is finite when the algebra $\mathcal{A}_F$ of coordinates on F is finite dimensional. We no longer require that this algebra is commutative.

We classified the irreducible $(\mathcal{A}, \mathcal{H}, J)$ and found out that the solutions fall into two classes. Let $\mathcal{A}_{\mathbb{C}}$ be the complex linear space generated by $\mathcal{A}$ in $\mathcal{L}(\mathcal{H})$, the algebra of operators in $\mathcal{H}$. By construction $\mathcal{A}_{\mathbb{C}}$ is a complex algebra and one only has two cases :

1. The center $Z\left(\mathcal{A}_{\mathbb{C}}\right)$ is $\mathbb{C}$, in which case $\mathcal{A}_{\mathbb{C}} = M_k(\mathbb{C})$ for some k.
2. The center $Z\left(\mathcal{A}_{\mathbb{C}}\right)$ is $\mathbb{C} \oplus \mathbb{C}$ and $\mathcal{A}_{\mathbb{C}} = M_k(\mathbb{C}) \oplus M_k(\mathbb{C})$ for some k.

Moreover the knowledge of $\mathcal{A}_{\mathbb{C}} = M_k(\mathbb{C})$ shows that $\mathcal{A}$ is either $M_k(\mathbb{C})$ (unitary case), $M_k(\mathbb{R})$ (real case) or, when $k = 2\ell$ is even, $M_\ell(\mathbb{H})$, where $\mathbb{H}$ is the field of quaternions (symplectic case). This first case is a minor variant of the Einstein-Yang-Mills case described above.

It turns out by studying their $\mathbb{Z}/2$ gradings γ, that these cases are incompatible with KO-dimension 6 which is only possible in case (2).

If one assumes that one is in the "symplectic–unitary" case and that the grading is given by a grading of the vector space over $\mathbb{H}$, one can show that the dimension of $\mathcal{H}$ which is $2k^2$ in case (2) is at least 2×16 while the simplest solution is given by the algebra $\mathcal{A} = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$. This is an important variant of the Einstein-Yang-Mills case because, as the center $Z(\mathcal{A}_{\mathbb{C}})$ is $\mathbb{C} \oplus \mathbb{C}$, the product of this finite geometry F by a manifold M appears, from the commutative standpoint, as two distinct copies of M.

We showed that requiring that these two copies of M stay a finite distance apart reduces the symmetries from the group SU(2)×SU(2)× SU(4) of inner automorphisms * to the symmetries $U(1) \times$ SU(2) $\times$ SU(3) of the Standard Model. This reduction of the gauge symmetry occurs because of the second kinematical condition $[[D,a],b] = 0$ which in the general case becomes :

$$[[D,a],b^0] = 0\,, \quad \forall\, a,b \in \mathcal{A}$$

∗. of the even part of the algebra

Spectral Model

Let M be a Riemannian spin 4-manifold and F the finite noncommutative geometry of KO-dimension 6 described above. Let $M \times F$ be endowed with the product metric.

1. The unimodular subgroup of the unitary group acting by the adjoint representation Ad(u) in $\mathcal{H}$ is the group of gauge transformations of SM.

2. The unimodular inner fluctuations of the metric give the gauge bosons of SM.

3. The full standard model (with neutrino mixing and seesaw mechanism) minimally coupled to Einstein gravity is given in Euclidean form by the action functional

$$S = \mathrm{Tr}(f(D_A/\Lambda)) + \frac{1}{2}\langle J\,\tilde{\xi}, D_A\,\tilde{\xi}\rangle\,, \quad \tilde{\xi} \in \mathcal{H}^+_{cl},$$

where D_A is the Dirac operator with the unimodular inner fluctuations.

Standard Model	Spectral Action
Higgs Boson	Inner metric$^{(0,1)}$
Gauge bosons	Inner metric$^{(1,0)}$
Fermion masses u, ν	Dirac$^{(0,1)}$ in $\uparrow$
CKM matrix Masses down	Dirac$^{(0,1)}$ in $(\downarrow 3)$
Lepton mixing Masses leptons e	Dirac$^{(0,1)}$ in $(\downarrow 1)$
Majorana mass matrix	Dirac$^{(0,1)}$ on $E_R \oplus J_F E_R$
Gauge couplings	Fixed at unification
Higgs scattering parameter	Fixed at unification
Tadpole constant	$-\mu_0^2 \, \lvert\mathbf{H}\rvert^2$

Noncommutative geometry was shown to provide a promising framework for unification of all fundamental interactions including gravity. Historically, the search to identify the structure of the noncommutative space followed the bottom-up approach where the known spectrum of the fermionic particles was used to determine the geometric data that defines the space. This bottom-up approach involved an interesting interplay with experiments. While at first the experimental evidence of neutrino oscillations contradicted the first attempt, it was realized several years later in 2006 that the obstruction to get neutrino oscillations was naturally eliminated by dropping the equality between the metric dimension of space-time (which is equal to 4 as far as we know) and its KO-dimension which is only defined modulo 8. When the latter is set equal to 2 modulo 8 (using the freedom to adjust the geometry of the finite space encoding the fine structure of space-time) everything works fine, the neutrino oscillations are there as well as the see-saw mechanism which appears for free as an unexpected bonus. Incidentally, this also solved the fermionic doubling problem by allowing a simultaneous Weyl-Majorana condition on the fermions to halve the degrees of freedom.

The second interplay with experiments occurred a bit later when it became clear that the mass of the Brout-Englert-Higgs boson would not comply with the restriction (that $m_H \succeq 170$ Gev) imposed by the validity of the Standard Model up to the unification scale.

New developments

We showed that the inconsistency between the spectral Standard Model and the experimental value of the Higgs mass is resolved by the presence of a real scalar field strongly coupled to the Higgs field. This scalar field was already present in the spectral model and we wrongly neglected it in our previous computations. It was shown recently by several authors, independently of the spectral approach, that such a strongly coupled scalar field stabilizes the Standard Model up to unification scale in spite of the low value of the Higgs mass. In our recent work, we show that the noncommutative neutral singlet modifies substantially the RG analysis, invalidates our previous prediction of Higgs mass in the range 160–180 Gev, and restores the consistency of the noncommutative geometric model with the low Higgs mass.

One lesson which we learned on that occasion is that we have to take all the fields of the noncommutative spectral model seriously, without making assumptions not backed up by valid analysis, especially because of the almost uniqueness of the Standard Model (SM) in the noncommutative setting.

The SM continues to conform to all experimental data. The question remains whether this model will continue to hold at much higher energies, or whether there is a unified theory whose low-energy limit is the SM. One indication that there must be a new higher scale that effects the low energy sector is the small mass of the neutrinos which is explained through the see-saw mechanism with a Majorana mass of at least of the order of 10^{11}Gev. In addition and as noted above, a scalar field which acquires a vev generating that mass scale can stabilize the Higgs coupling and prevent it from becoming negative at higher energies and thus make it consistent with the low Higgs mass of 126 Gev. Another indication of the need to modify the SM at high energies is the failure (by few percent) of the three gauge couplings to be unified at some high scale which indicates that it may be necessary to add other matter couplings to change the slopes of the running of the RG equations.

This leads us to address the issue of the breaking from the natural algebra $\mathcal{A}$ which results from the classification of irreducible finite geometries of KO-dimension 6 (modulo 8), to the algebra corresponding to the SM. This breaking was effected using the requirement of the first order condition on the Dirac operator. The first order condition is the requirement that the Dirac operator is a derivation of the algebra $\mathcal{A}$ into the commutant of $\hat{\mathcal{A}} = J\mathcal{A}J^{-1}$ where J is the charge conjugation operator. This in turn guarantees the gauge invariance and linearity of the inner fluctuations under the action of the gauge group given by the unitaries $U = uJuJ^{-1}$ for any unitary $u \in \mathcal{A}$. This condition was used as a mathematical requirement to select the maximal subalgebra

$$\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \subset \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$$

which is compatible with the first order condition and is the main reason behind the unique selection of the SM.

The existence of examples of noncommutative spaces where the first order condition is not satisfied such as quantum groups and quantum spheres provides a motive to remove this condition from the classification of noncommutative spaces compatible with unification. This study was undertaken in a companion paper where it was shown that in the general case the inner fluctuations of D with respect to inner automorphisms of the form $U = u\, Ju\, J^{-1}$ are given by

$$D_A = D + A_{(1)} + \widetilde{A}_{(1)} + A_{(2)}$$

where

$$\begin{aligned}
A_{(1)} &= \sum_i a_i\, [D, b_i] \\
\widetilde{A}_{(1)} &= \sum_i \widehat{a}_i \left[D, \widehat{b}_i\right], \qquad \widehat{a}_i = J a_i J^{-1}, \quad \widehat{b}_i = J b_i J^{-1} \\
A_{(2)} &= \sum_{i,j} \widehat{a}_i a_j \left[\left[D, b_j\right], \widehat{b}_i\right] = \sum_{i,j} \widehat{a}_i \left[A_{(1)}, \widehat{b}_i\right].
\end{aligned}$$

Clearly $A_{(2)}$ which depends quadratically on the fields in $A_{(1)}$ vanishes when the first order condition is satisfied.

Our point of departure is that one can extend inner fluctuations to the general case, *i.e.* without assuming the order one condition. It suffices to add a quadratic term which only depends upon the universal 1-form $\omega \in \Omega^1(\mathcal{A})$ to the formula and one restores in this way,

– The gauge invariance under the unitaries $U = uJuJ^{-1}$
– The fact that inner fluctuations are transitive, *i.e.* that inner fluctuations of inner fluctuations are themselves inner fluctuations.

We show moreover that the resulting inner fluctuations come from the action on operators in Hilbert space of a semi-group Pert$(\mathcal{A})$ of *inner perturbations* which only depends on the involutive algebra $\mathcal{A}$ and extends the unitary group of $\mathcal{A}$. This opens up two areas of investigation, the first is mathematical and the second is directly related to particle physics and model building :

1. Investigate the inner fluctuations for noncommutative spaces such as quantum groups and quantum spheres.
2. Compute the spectral action and inner fluctuations for the model involving the full symmetry algebra $\mathbb{H}\oplus\mathbb{H}\oplus M_4(\mathbb{C})$ before the breaking to the Standard Model algebra.

(i) The following map η is a surjection

$$\eta : \{\sum a_j \otimes b_j^{\mathrm{op}} \in \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \mid \sum a_j b_j = 1\} \to \Omega^1(\mathcal{A}),$$

$$\eta(\sum a_j \otimes b_j^{\mathrm{op}}) = \sum a_j \delta(b_j).$$

(ii) One has

$$\eta\left(\sum b_j^* \otimes a_j^{*\mathrm{op}}\right) = \left(\eta\left(\sum a_j \otimes b_j^{\mathrm{op}}\right)\right)^*$$

(iii) One has, for any unitary $u \in \mathcal{A}$,

$$\eta\left(\sum u a_j \otimes (b_j u^*)^{\mathrm{op}}\right) = \gamma_u\left(\eta\left(\sum a_j \otimes b_j^{\mathrm{op}}\right)\right)$$

where γ_u is the gauge transformation of potentials.

(i) Let $A = \sum a_j \otimes b_j^{\mathrm{op}} \in \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ normalized by the condition $\sum a_j b_j = 1$. Then the operator $D' = D(\eta(A))$ is equal to the inner fluctuation of D with respect to the algebra $\mathcal{A} \otimes \widehat{\mathcal{A}}$ and the 1-form $\eta(A \otimes \widehat{A})$, that is

$$D' = D + \sum a_i \hat{a}_j [D, b_i \hat{b}_j]$$

(ii) An inner fluctuation of an inner fluctuation of D is still an inner fluctuation of D, and more precisely one has, with A and A' normalized elements of $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ as above,

$$(D(\eta(A)))\,(\eta(A')) = D(\eta(A'A))$$

where the product $A'A$ is taken in the tensor product algebra $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$.

(i) The self-adjoint normalized elements of $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ form a semi-group Pert$(\mathcal{A})$ under multiplication.

(ii) The transitivity of inner fluctuations (*i.e.* the fact that inner fluctuations of inner fluctuations are inner fluctuations) corresponds to the semi-group law in the semi-group Pert$(\mathcal{A})$.

(iii) The semi-group Pert$(\mathcal{A})$ acts on real spectral triples through the homomorphism

$$\mu : \mathrm{Pert}(\mathcal{A}) \to \mathrm{Pert}(\mathcal{A} \otimes \hat{\mathcal{A}})$$

given by

$$A \in \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \mapsto \mu(A) = A \otimes \hat{A} \in \left(\mathcal{A} \otimes \hat{\mathcal{A}}\right) \otimes \left(\mathcal{A} \otimes \hat{\mathcal{A}}\right)^{\mathrm{op}}$$

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Le Théâtre quantique
$e^{iS/\hbar}$
$\delta : \mathbb{R} \longrightarrow \mathrm{Out}(M)$
$\sqrt{2}\psi = |\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle$
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