Group Actions on Kirchberg Algebras

Masaki Izumi

Graduate School of Science, Kyoto University

July 9, 2013 at Rome

Joint work with Hiroki Matui

In the case of AFD factors

"Very outer" actions of a discrete amenable group G on an AFD factor M are completely classified by local invariants.

Theorem (Connes, Jones, Ocneanu, Takesaki, Sutherland, Kawahigashi, Katayama)

Centrally free actions α of G on M are completely classified by $\mathrm{mod}\,(\alpha_g)$, *where*

$$\mathrm{mod}\ : \mathrm{Aut}(M) \to \mathrm{Aut}(F^M)$$

is the Connes-Takesaki module.

The classification of non centrally free actions requires global (cohomological) invariants, called the characteristic invariant.

Nakamura's theorem

Outer $\mathbb{Z}$-actions on a Kirchberg algebras are completely classified by local invariants.

Theorem (Nakamura 1999)

Let A be Kirchberg algebra, and let $\alpha, \beta \in \mathrm{Aut}(A)$.
If α^n and β^n are outer for all $n \in \mathbb{Z} \setminus \{0\}$, the following conditions are equivalent:
(1) $KK(\alpha) = KK(\beta)$,
(2) $\exists \gamma \in \mathrm{Aut}(A)$, $\exists u \in U(A)$ *s.t.* $KK(\gamma) = KK(\mathrm{id})$, *and*

$$\mathrm{Ad}\, u \circ \alpha = \gamma \circ \beta \circ \gamma^{-1}.$$

Goal

Goal To classify "very outer" actions of "nice" discrete amenable groups G on classifiable nuclear C^*-algebras A.

Question Are local invariants sufficient for general G?

Answer No!

Reason The interplay between G (or its classifying space BG) and the topology of $\mathrm{Aut}(A)$ (or $\mathrm{Aut}(A \otimes \mathbb{K})$) matters.

For G with low cohomological dimension, there is a good chance to classify G actions.

Kirchberg algebras

Definition

- A unital C^*-algebra A is purely infinite if $\forall a \in A_+ \setminus \{0\}$, $\exists x \in A$ such that $1 = x^*ax$.
- A Kirchberg algebra is a purely infinite, simple, nuclear, separable C^*-algebra.

Theorem (Kirchberg, Phillips)

Kirchberg algebras are completely classified by KK-theory.
Kirchberg algebras A and B satisfying UCT are isomorphic iff

$$(K_0(A), [1_A], K_1(A)) \cong (K_1(B), [1_B], K_1(B)).$$

Let A and B be Kirchberg algebras, and $\rho_1, \rho_2 \in \mathrm{Hom}(A, B)$.
$KK(\rho_1) = KK(\rho_2) \Leftrightarrow \rho_1, \rho_2$ are asymptotically unitarily equivalent, i.e. there exists a continuous path $\{u(t)\}_{t \geq 0}$ in $U(B)$ s.t.

$$\lim_{t \to \infty} \| \mathrm{Ad}\, u(t) \circ \rho_2(x) - \rho_1(x) \| = 0, \ \forall x \in A.$$

Cuntz algebras

Example

Cuntz algebra $\mathcal{O}_n$ is the universal C^*-algebra generated by isometries $S_1, S_2, \cdots, S_n$ satisfying $S_i^* S_j = \delta_{ij} 1$, and $\sum_{i=1}^{n} S_i S_i^* = 1$ if $n < \infty$.

$\mathcal{O}_n$ is a Kirchberg algebra.

$(K_0(\mathcal{O}_n), [1_{\mathcal{O}_n}], K_1(\mathcal{O}_n)) \cong (\mathbb{Z}/(n-1)\mathbb{Z}, 1, \{0\})$ if $n < \infty$,
$(K_0(\mathcal{O}_\infty), [1_{\mathcal{O}_\infty}], K_1(\mathcal{O}_\infty)) \cong (\mathbb{Z}, 1, \{0\})$.

$\mathcal{O}_2 \overset{KK}{\sim} \{0\}$,
$A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ for any Kirchberg algebra A.

$\mathcal{O}_\infty \overset{KK}{\sim} \mathbb{C}$,
$A \otimes \mathcal{O}_\infty \cong A$ for any Kirchberg algebra A.

Equivalence relations

Definition

Let α, β be actions of a discrete group G on a C^*-algebra A.

- A map $u : G \to U(A)$ is an α-cocycle if $u_{gh} = u_g \alpha_g(u_h)$. $\mathrm{Ad}\, u_g \circ \alpha_g$ is a G-action too, called a cocycle perturbation of α.
- α and β are cocycle conjugate if there exist an α-cocycle u and $\gamma \in \mathrm{Aut}(A)$ satisfying

$$\mathrm{Ad}\, u_g \circ \alpha_g = \gamma \circ \beta_g \circ \gamma^{-1}.$$

- If moreover γ can be chosen to satisfy $KK(\gamma) = KK(\mathrm{id})$, we say that α are β are KK-trivially cocycle conjugate.

"Classification" always means up to KK-trivial cocycle conjugacy.

$KK(\alpha_g)$ is an invariant for a KK-trivial cocycle conjugacy class.

Poly-$\mathbb{Z}$ groups

Definition

A discrete group G is poly-$\mathbb{Z}$ if there exists normal series

$$\{e\} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G,$$

such that $G_{i+1}/G_i \cong \mathbb{Z}$.
The number $h(G) = n$ is said to be the Hirsch length of G.

Every finitely generated torsion free nilpotent group is poly-$\mathbb{Z}$.
Every cocompact lattice of a simply connected solvable Lie group is poly-$\mathbb{Z}$.

Example

$\mathbb{Z}^n$,
$< a, b | \, aba^{-1}b = 1 >= \pi_1(\text{Klein bottle})$,
$\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} ; \; a, b, c \in \mathbb{Z} \}$: Discrete Heisenberg group,

Uniqueness

Theorem (I.-Matui)

Let G be a poly-$\mathbb{Z}$ group.
Let A be either $\mathcal{O}_2$, $\mathcal{O}_\infty$, or $\mathcal{O}_\infty \otimes B$ with UHF algebra B satisfying $B \otimes B \cong B$.
Then there exists a unique KK-trivial cocycle conjugacy class of outer G-actions on A.

A is called strongly self-absorbing.
$\pi_n(\mathrm{Aut}(A)) = \{0\}$ for all $n \geq 0$ (Dadarlat 2007).

Fix an outer action μ^G of G on $\mathcal{O}_\infty$.

Theorem (I.-Matui)

For any outer action α of a poly-$\mathbb{Z}$ group G on a Kirchberg algebra A, α is cocycle conjugate to $\alpha \otimes \mu^G$ on $A \otimes \mathcal{O}_\infty$.

Classification

In what follows A is always a Kirchberg algebra.

Theorem (I.-Matui)

Outer actions of a poly-$\mathbb{Z}$ group G with $h(G) \leq 3$ on a A is classifiable. The number of KK-trivially cocycle conjugacy classes is bounded by

$$\begin{aligned} &\#\mathrm{Hom}(G, KK(A,A)_u^{-1}) \\ &\times \#H^2(G, \pi_1(\mathrm{Aut}(A \otimes \mathbb{K})_0)) \times \#H^3(G, \pi_2(\mathrm{Aut}(A \otimes \mathbb{K})_0)), \end{aligned}$$

where

$$KK(A,A)_u^{-1} = \{x \in KK(A,A)^{-1};\ [1_A]\hat{\otimes} x = [1_A] \in K_0(A)\}.$$

$\pi_n(\mathrm{Aut}(A \otimes \mathbb{K})_0) \cong KK^n(A,A)$ (Dadarlat 2007).

Classification (continued)

Theorem (I. Matui)

Let G be a poly-Z group with $h(G) \leq 3$, and $2 \leq n < \infty$.
There exist exactly $\#H^2(G, \mathbb{Z}/(n-1)\mathbb{Z})$ outer actions of G on $\mathcal{O}_n$.

	Outer actions on $\mathcal{O}_n$
$\mathbb{Z}$	1
$\mathbb{Z}^2$	$n-1$
$\mathbb{Z}^3$	$(n-1)^3$
$\pi_1(\text{Klein bottle})$	1 for even n and 2 for odd n
Discrete Heisenberg group	$(n-1)^2$

There exist exactly $\#H^2(G, \mathbb{Z}/(n-1)\mathbb{Z}) \times \#H^3(G, \mathbb{Z}/(n-1)\mathbb{Z})$ outer cocycle actions of G on $\mathcal{O}_n$.

A necessary condition

Lemma

Let α, β be actions of a discrete group G on A.
If α and β are KK-trivially cocycle conjugate, then β is continuously approximated by cocycle perturbations of α.

Proof.

$\exists$ α-cocycle $\{u_g\}_{g\in G}$, $\exists$ continuous family $\{v(t)\}_{t\geq 0}$ in $U(A)$ s.t.

$$\mathrm{Ad}\, u_g \circ \alpha_g = \gamma \circ \beta_g \circ \gamma^{-1}, \quad \gamma = \lim_{t\to\infty} \mathrm{Ad}\, v(t).$$

Set $a_g(t) = v(t)^* u_g \alpha_g(v(t))$.
$\{a_g(t)\}_{g\in G}$ is an α-cocycle for each t and

$$\lim_{t\to\infty} \mathrm{Ad}\, a_g(t) \circ \alpha_g = \beta_g.$$

$\square$

Sufficiency

Theorem (I.-Matui)

Let α, β be outer actions of a poly-$\mathbb{Z}$ group G on A.
If there exist continuous families of unitaries $u_g(t)$ in A satisfying

$$\lim_{t\to\infty} \mathrm{Ad}\, u_g(t) \circ \alpha_g = \beta_g,$$

$$\lim_{t\to\infty} \|u_g(t)\alpha_g(u_h(t)) - u_{gh}(t)\| = 0,$$

then α and β are KK-trivially cocycle conjugate.

$A^\flat = C_b([0,\infty), A)/C_0([0,\infty), A)$.

If $\exists$ α-cocycle $\{u_g\}_{g\in G}$ in $U(A^\flat)$ satisfying $\mathrm{Ad}\, u_g \circ \alpha_g(x) = \beta_g(x)$ for any $x \in A$, then α and β are KK-trivially cocycle conjugate.

Difficulty in finite group actions

Let $\alpha \in \mathrm{Aut}(\mathcal{O}_2)$ be the flip automorphism $\alpha(S_1) = S_2$, $\alpha(S_2) = S_1$.
Then $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2 \cong \mathcal{O}_2$.

Theorem (I. 2004)

For any uniquely 2-divisible countable abelian groups M_0, M_1, $\exists$ an outer $\mathbb{Z}_2$-action β on the Cuntz algebra $\mathcal{O}_2$ s.t.

$$K_*(\mathcal{O}_2 \rtimes_\beta \mathbb{Z}_2) = M_*, \quad * = 0, 1.$$

Moreover, $\exists$ a continuous family $\{u(t)\}_{t \geq 0}$ in $U(\mathcal{O}_2)$ s.t.

$$u(t)\alpha(u(t)) = 1,$$

$$\beta(x) = \lim_{t \to \infty} \mathrm{Ad}\, u(t) \circ \alpha(x), \quad \forall x \in \mathcal{O}_2.$$

Primary obstruction

Let $A^\flat = C_b([0,\infty), A)/C_0([0,\infty), A)$, $A_\flat = A^\flat \cap A'$.

Let α, β be outer actions of a discrete group G on A satisfying $KK(\alpha_g) = KK(\beta_g)$.
Choose $u_g \in U(A^\flat)$ satisfying $\mathrm{Ad}\, u_g \circ \alpha_g(x) = \beta_g(x)$ for any $x \in A$.

Set $w_{g,h} = u_g \alpha_g(u_h) u_{gh}^* \in U(A_\flat)$. Set $\sigma_g = \mathrm{Ad}\, u_g \circ \alpha_g|_{A_\flat}$.
(σ, w) is a cocycle action of G on $A_\flat$.

(σ, w) is equivalent to a genuine action
$\Leftrightarrow$ $\{u_g\}_{g\in G}$ can be chosen to from an α-cocycle.

$\mathfrak{o}^2(\alpha,\beta) = [(K_1(w_{g,h}))_{g,h\in G}] \in H^2(G, K_1(A_\flat))$ does not depend on the choice of $\{u_g\}_{g\in G}$.
We call $\mathfrak{o}^2(\alpha,\beta)$ primary obstruction, which is an obstruction for α and β to be KK-trivially cocycle conjugate.

Higher obstruction

When $\mathfrak{o}^2(\alpha, \beta) = 0$, we can choose $\{u_g\}_{g \in G}$ so that $w_{g,h} \in U(A_\flat)_0$.

Choose a continuous path $\{\tilde{w}_{g,h}(t)\}_{t \in [0,1]}$ from 1 to $w_{g,h}$ in $U(A_\flat)_0$. Then

$$K_1(\sigma_g(\tilde{w}_{h,k})\tilde{w}_{g,hk}\tilde{w}^*_{gh,k}\tilde{w}_{g,h}) \in K_1(SA_\flat) = K_0(A_\flat).$$

We can define $\mathfrak{o}^3(\alpha, \beta, u) \in H^3(G, K_0(A_\flat))$ by the cohomology class of it, which does not depends on the choice of $\tilde{w}_{g,h}(t)$.
$\mathfrak{o}^3(\alpha, \beta, u)$ may depend on the choice of $\{u_g\}_{g \in G}$.

Theorem (I.-Matui)

For each finite CW-complex X, there exists an isomorphism from $[X, U(A_\flat)]_0$ onto $[X, \mathrm{Map}(S^1, \mathrm{Aut}(A \otimes \mathbb{K}))]_0$, which is natural in X. In particular, $K_n(A_\flat) \cong \pi_n(\mathrm{Aut}(A \otimes \mathbb{K})_0)$.

Recall $\pi_n(\mathrm{Aut}(A \otimes \mathbb{K})_0) \cong KK^n(A, A)$.

Classification by obstructions

Theorem (I.-Matui)

Let α, β be outer actions of a poly-$\mathbb{Z}$ group on A.
Assume $KK(\alpha_g) = KK(\beta_g)$ for any $g \in G$.

(1) *Assume $h(G) = 2$.*
α and β are KK-trivially cocycle conjugate if and only if $\mathfrak{o}^2(\alpha, \beta) = 0$.

(2) *Assume $h(G) = 3$.*
α and β are KK-trivially cocycle conjugate if and only if $\mathfrak{o}^2(\alpha, \beta) = 0$ and $\mathfrak{o}^3(\alpha, \beta, u) = 0$ for some choice of $\{u_g\}_{g \in G}$.

Conjecture

Let BG be the classifying space of a poly-$\mathbb{Z}$ group G, and let EG be its universal covering space,
e.g, $G = \mathbb{Z}^N$, $EG = \mathbb{R}^N$, $BG = \mathbb{T}^N$.

For a G-action α on A, we denote by $\mathcal{P}_\alpha$ the quotient space of $EG \times \mathrm{Aut}(A)$ by the equivalence relation $(x \cdot g, \gamma) \sim (x, \alpha_g \circ \gamma)$.
$\mathcal{P}_\alpha$ is a principal $\mathrm{Aut}(A)$-bundle over BG.

We define $\mathcal{P}^s_\alpha$ in the same way by replacing α_g by $\alpha_g \otimes \mathrm{Ad}\,\rho_g$, and $\mathrm{Aut}(A)$ with the group generated by

$$\{\gamma \otimes \mathrm{id}_{\mathbb{K}} \in \mathrm{Aut}(A \otimes \mathbb{K});\ \gamma \in \mathrm{Aut}(A)\} \cup \mathrm{Inn}(A \otimes \mathbb{K}).$$

Conjecture

For two outer G-actions α, β on A, TFAE:
(1) α *and* β *are* KK*-trivially cocycle conjugate.*
(2) $\mathcal{P}^s_\alpha$ *and* $\mathcal{P}^s_\beta$ *are isomorphic by a base point preserving map.*

Conjecture (continued)

The primary (resp. higher) obstruction for the existence of a base point preserving isomorphism between $\mathcal{P}^s_\alpha$ and $\mathcal{P}^s_\beta$ can be identified with $\mathfrak{o}^2(\alpha, \beta)$ (resp. $\mathfrak{o}^3(\alpha, \beta, u)$).

Corollary

Conjecture is true for $h(G) \leq 3$.

When A is a strongly self-absorbing, e.g. $\mathcal{O}_2$, $\mathcal{O}_\infty$, we have $\pi_n(\mathrm{Aut}(A)) = \{0\}$ for all n, and so $\mathcal{P}_\alpha$ is a trivial bundle.

Corollary

Conjecture is true for strongly self-absorbing A.