Algebraic theory of integrable PDE

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Evolution equation is a PDE of the form

$$\frac{du}{dt} = P(u, u', \dots, u^{(n)}), \tag{1}$$

where $u = \begin{pmatrix} u_1 \\ \vdots \\ u_\ell \end{pmatrix}$, $u_i = u_i(t, x)$ is a function in one independent variable x, and t (time) is a parameter;

$P = \begin{pmatrix} P_1 \\ \vdots \\ P_\ell \end{pmatrix} \in V^\ell$, V algebra of "differential functions".

This equation is called *compatible* with another evolution equation

$$\frac{du}{dt_1} = Q(u, u', \dots, u^{(m)})$$

if "the corresponding flows commute":

$$\frac{d}{dt}\frac{d}{dt_1}u = \frac{d}{dt_1}\frac{d}{dt}u.$$

Compute the LHS using the chain rule:

$$\frac{d}{dt}\frac{d}{dt_1}Q(u,u',\dots,u^{(m)}) = \sum_{i;\ n\in\mathbb{Z}_+}\frac{\partial Q}{\partial u_i^{(n)}}\partial^n P_i = X_P Q,$$

where

$$\partial = \frac{d}{dx}$$

is the *total derivative*, and

$$X_P = \sum_{i;\ n\in\mathbb{Z}_+}(\partial^n P_i)\frac{\partial}{\partial u_i^{(n)}}$$

is the *evolutionary vector field* with characteristic $P \in V^\ell$.

Hence,

$$\left[\frac{d}{dt},\frac{d}{dt_1}\right]u = [X_P, X_Q] = X_{[P,Q]},$$

where

$$[P,Q] = X_P Q - X_Q P \tag{2}$$

is a Lie algebra bracket on V^ℓ.

Thus, equations $\frac{dt}{du} = P$, $\frac{du}{dt_1} = Q$ are compatible iff the corresponding evolutionary vector fields commute.

Evolution equation is called *integrable* if it can be included in an infinite hierarchy of linearly independent compatible evolution equations:

$$\frac{du}{dt_n} = P_n, \ \ [P_m, P_n] = 0, \ \ m, n \in \mathbb{Z}_+,$$

called an *integrable hierarchy*.

Thus, classification of integrable evolution equations = classification of infinite-dimensional (maximal) abelian subalgebras $\mathcal{L}$ in the Lie algebra of evolutionary vector fields V^ℓ with the bracket (2).

Trivial examples of integrable hierarchies:

1. linear: $u_{t_n} = u^{(n)}$,
   since $X_{u^{(m)}} u^{(n)} = u^{(m+n)}$
2. dispersionless: $u_{t_f} = f(u)u'$,
   since $X_{f(u)u'}(g(u)u') = \left(f\frac{dg}{du} + g\frac{df}{du}\right) u'^2 + fgu''$.

Nontrivial examples of integrable hierarchies:

$$
\begin{array}{lcll}
u_t & = & u'' + uu' & \text{(Burgers)} \\
u_t & = & u''' + uu' & (KdV) \\
u_t & = & u''' + u^2 u' & (mKdV) \\
u_t & = & u''' + u'^2 & (pKdV) \\
u_t & = & u''' + u'^3 & (LKdV) \\
u_t & = & \underbrace{u''' - \dfrac{3u''^2}{2u'}}_{\text{Schwarz KdV}} + \frac{h(u)}{u'} & \text{(Krichever–Novikov)}
\end{array}
$$

$h(u)$ polynomial of degree at most 4.

Shabat, Sokolov, Mikhailov,..., Meshkov Theorem. Up to automorphism of the algebra of differential functions, there are only nine more integrable equations of the form $u_t = u''' + f(u, u', u'')$.

Folklore Conjecture. Any order ≥ 7 integrable evolution equation in one function u is contained in the hierarchy of a non-trivial integrable equation of order ≤ 5. In other words, any maximal infinite-dimensional subalgebra of V with bracket (2) contains a non-central element of order ≤ 5.

There are partial classificational results on 2-component equations, the most famous among them is the non-linear Schrödinger:

$$\begin{cases} u_t &= v'' + 2v(u^2+v^2) \\ v_t &= -u'' - 2u(u^2+v^2) \end{cases}.$$

I shall now discuss the other part of the problem: how to prove integrability. But first we have to answer the usually neglected question:

What is a differential fuction $f \in V$?

An *algebra of differential functions* is a differential algebra V with the derivation ∂ (total derivative), endowed with commuting derivations

$$\frac{\partial}{\partial u_i^{(n)}}, \ i = 1, \ldots, \ell\,; \, n \in \mathbb{Z}_+,$$

subject to two axioms:

1 $\frac{\partial}{\partial u_i^{(n)}} f = 0$ for all but finite number of i, n.

2 $\left[\frac{\partial}{\partial u_i^{(n)}}, \partial\right] = \frac{\partial}{\partial u_i^{(n-1)}}$ (basic identity).

Axiom 1 is needed, otherwise $X_P Q$ is divergent.

Axiom 2 is satisfied for the main example, the algebra of differential polynomials:

$$V = \mathbb{F}[u_i^{(n)} | i = 1, \ldots, \ell\,; \, n \in \mathbb{Z}_+]$$
$$\partial u_i^{(n)} = u_i^{(n+1)}.$$

Arbitrary V is its extension, for example, for KN we need to invert u'.

Note: ∂^{-1} cannot be defined if we want both axioms to hold!

S.-S. Chern. In life both men and women are important. Likewise in geometry both vector fields and differential forms are important.

In our theory vector fields are evolutionary vector fields

$$X_P \ (P \in V^\ell).$$

They commute with $\partial = X_{u'}$. This tells us how to define *variational differential forms*.

Ordinary differential forms (dual to all vector fields) are

$$\omega = \sum f_{i_1,\ldots,i_k}^{n,\ldots,n_k}\, du_{i_1}^{(u_1)} \wedge \cdots \wedge du_{i_k}^{(n_k)}$$

with the usual de Rham differential d:

$$\widetilde{\Omega}^0 = V \xrightarrow{d} \widetilde{\Omega}^1 \xrightarrow{d} \widetilde{\Omega}^2 \to \cdots$$

and derivation $\partial : \partial(du_i^{(n)}) = du_i^{(n+1)}$.

Axiom 2. of V (the basic identity) is equivalent to the property that ∂ commutes with d. Therefore we can define the *variational complex* by letting

$$\Omega^k = \widetilde{\Omega}^k / \partial\widetilde{\Omega}^k :$$

$$V/\partial V \xrightarrow{d} \widetilde{\Omega}^1/\partial\widetilde{\Omega}^1 \xrightarrow{d} \widetilde{\Omega}^2/\partial\widetilde{\Omega}^2 \xrightarrow{d} \ldots$$

Here $V/\partial V$ is the space (not algebra any more) of *local functionals*, the universal space where we can perform integration by parts. Now we can describe the variational complex more explicitely:

$$
\begin{aligned}
V/\partial V \to V^{\oplus \ell} \to & \text{ skew-adjoint matrix} \\
& \text{differential operators on } V^{\ell} \to \ldots \\
\int f \mapsto & \frac{\delta \int f}{\delta u} \\
F \mapsto & D_F - D_F^*
\end{aligned}
$$

where
$$\frac{\delta}{\delta u} = \left(\frac{\delta}{\delta u_j}\right)_j, \quad \frac{\delta f}{\delta u_i} = \sum_{n \in \mathbb{Z}_+} (-\partial)^n \frac{\partial f}{\partial u_i^{(n)}}$$
is the variational derivative;
$$(D_F)_{ij} = \sum_{n \in \mathbb{Z}_+} \frac{\partial F_i}{\partial u_j^{(n)}} \partial^n$$
is the Frechet derivative.

Note that

(a) $\frac{\delta}{\delta u} \circ \partial = 0$ ($\Leftrightarrow$ axiom 2) (Euler)

(b) $D_{\frac{\delta f}{\delta u}}$ is self-adjoint (Helmholtz), is the condition on $F \in V^{\oplus \ell}$ to be a variational derivative (exact 1-form is closed)

Theorem. Let

$$V_{m,i} = \{f \in V | \frac{\partial}{\partial u_j^{(n)}} f = 0\,, \quad (n,j) > (m,i)\}$$

and suppose that $\frac{\partial}{\partial u_i^{(m)}} V_{m,i} = V_{m,i}$. Then the variational complex is exact. One can always embed V in a larger algebra of differential functions $\widetilde{V}$ s.t. the variational complex becomes exact.

Note that we a have a non-degenerate *pairing* between the space of evolutionary vector fields $= V^{\ell}$ and the space of variational 1-forms $\Omega^1 = V^{\oplus \ell}$, induced from the usual pairing of vector fields with differential 1-forms:

(3) $$(X_P | \omega_Q) = (P|Q) := \int P \cdot Q \in V/\partial V\,.$$

An effective way of constructing an integrable equation is to use Poisson structures. What is a local (or non-local) Poisson structure on V?

Physicists define it by the following formula:

(4)

$$\{u_i(x), u_j(y)\} = H_{ij}(u(y), u'(y), \ldots, u^{(n)}(y); \partial/\partial y)\delta(x-y)\,,$$

where $\int f(y)\delta(x-y) = f(x)$ and $H = (H_{ij})$ is an $\ell \times \ell$ matrix differential (or pseudo-differential) operator, whose coefficients are functions in $u, u', \ldots, u^{(n)}$.

Extending this formula (4) by Leibniz's rule and bilinearity to $f, g \in V$, we obtain

(5)

$$\{f(x), g(y)\} = \sum_{i,j} \sum_{m,n \in \mathbb{Z}_+} \frac{\partial f(x)}{\partial u_i^{(m)}} \frac{\partial g(y)}{\partial u_j^{(n)}} \partial_x^m \partial_y^n \{u_i(x), u_j(y)\}.$$

Integrating (5) by parts in x, we obtain (for $g = u_j$):

$$\{\int f, u\}_H = H \frac{\delta}{\delta u} \int f\,. \tag{6}$$

Integrating (5) by parts in x and in y, we obtain:

$$\{\int f, \int g\}_H = \int \frac{\delta \int g}{\delta u} \cdot H(\partial) \frac{\delta \int f}{\delta u}\,. \tag{7}$$

Definition (a) An $\ell \times \ell$ matrix differential operator H is called a (local) *Poisson structure* on V if (7) is a Lie algebra bracket on $V/\partial V$. This happens iff $H^* = -H$ and $[H, H]$ (Schouten bracket) $= 0$.

(b) Given a Poisson structure H on an algebra of differential functions V and a local functional $\int h$ (Hamiltonian), the corresponding *Hamiltonian evolution equation* is

$$\frac{du}{dt} = \{\int h, u\}_H \tag{8}$$

(the corresponding evolutionary vector field is $X_{H\frac{\delta \int h}{\delta u}}$).

(c) Two local functionals are in *involution* if their commutator (7) is zero.

Remark. The map $V/\partial V \to$ Lie algebra of evolutionary vector fields V^ℓ given by

$$\int f \mapsto X_{H\frac{\delta \int f}{\delta u}}$$

is a Lie algebra homomorphism. In particular, local functionals in involution correspond to commuting evolutionary vector fields.

Corollary. If $\int h$ is contained in an infinite-dimensional abelian subalgebra of the Lie algebra $(V/\partial V, \{\ ,\ \}_H)$ and dim Ker $H < \infty$ (i.e. H non-degenerate), then equation (8) is integrable.

An alternative approach is to apply the Fourier transform $\int dxe^{\lambda(x-y)}$. to both sides of (5). Denoting $\{f_\lambda g\} = \int dxe^{\lambda(x-y)}\{f(x), f(y)\}$, we get the Master Formula:

$$\text{(9)} \qquad \{f_\lambda g\} = \sum_{i,j=1}^{\ell} \sum_{m,n\in\mathbb{Z}_+} \frac{\partial g}{\partial u_j^{(n)}}(\lambda+\partial)^n H_{ji}(-\lambda-\partial)^m \frac{\partial f}{\partial u_i^{(m)}}\,.$$

This λ-bracket satisfies:

(i) (Leibniz rules) $\{f_\lambda gh\} = g\{f_\lambda h\} + h\{f_\lambda g\}$; $\{fg_\lambda h\} = \{f_{\lambda+\partial} g\}_\to h + \{f_{\lambda+\partial} h\}_\to g$;

(ii) (sesquilinearity) $\{\partial f_\lambda g\} = -\lambda\{f_\lambda g\}$, $\{f_\lambda \partial g\} = (\lambda+\partial)\{f_\lambda g\}$.

Theorem. (a) The bracket (7) is a Lie algebra bracket iff:

(iii) (skewcommutativity) $\{g_\lambda f\} = -\{f_{-\lambda-\partial} g\}$,

(iv) (Jacobi identity) $\{f_\lambda\{g_\mu h\}\} - \{g_\mu\{f_\lambda h\}\} = \{\{f_\lambda g\}_{\lambda+\mu} h\}$.

(b) It suffices to check skewcommutativity of any pair (u_i, u_j) and Jacobi identity for any triple (u_i, u_j, u_k).

Definition. (a) A $\mathbb{F}[\partial]$-module R is called a *Lie conformal algebra* if $\{R_\lambda R\} \subset R[\lambda]$ and (ii), (iii), (iv) hold.

(b) A unital differential algebra (V, ∂) is called a (local) *Poisson vertex algebra* (PVA) if $\{V_\lambda V\} \subset V[\lambda]$ and (i)–(iv) hold.

(c) If the λ-bracket is given by the Master formula, and it is a PVA, the (skewadjoint) differential operator $H = (H_{ij})$ is called a (local) *Poisson structure*.

Examples. $H = \partial$ (GFZ structure) $\{u_\lambda u\} = \lambda$

$H = c\partial^3 + 2u\partial + u'$ (Virasoro–Magri structure) $\{u_\lambda u\} = 2u\lambda + u' + c\lambda^3$.

How to extend these notions to the non-local case (i.e. $H(\partial)$ is a pseudodifferential operator?). In this case we see from (9) that

$$\{V_\lambda V\} \subset V((\lambda^{-1}))\,.$$

It is easy to interpret the identities (i)–(iii): expand in positive powers of ∂ each time when we encounter $\frac{1}{(\lambda+\partial)^n}$. However, in order for the Jacobi identity to make sense we must impose *admissibility* property:

$$\{f_\lambda\{g_\mu h\}\} \subset V[[\lambda^{-1},\mu^{-1},(\lambda+\mu)^{-1}]][\lambda,\mu]\,.$$

Proposition. The λ-bracket (9), given by the Master Formula is admissible provided that $H(\partial)$ is a *rational* pseudodifferential operator, i.e. it is contained in the subalgebra of the algebra of pseudodifferential operators $V((\partial^{-1}))$, generated by differential operators and their inverses.

Then our basic definitions extend to the non-local case: non-local Lie conformal algebra, non-local PVA, non-local Poisson structure.

Examples: $H = \partial^{-1}$
$H = u'\partial^{-1} \circ u'$ (Sokolov)
$H = \partial^{-1} \cdot u'\partial^{-1} \circ u'\partial^{-1}$ (Dorfman)

$H = \partial I_2 + \begin{pmatrix} v\partial^{-1} \circ v & -v\partial^{-1} \circ u \\ -u\partial^{-1} \circ v & u\partial^{-1} \circ u \end{pmatrix}$ (Magri: non-local Poisson structure for NLS)

A theory of rational pseudodifferential operators.

Let (V, ∂) be a unital differential algebra, assume V is a domain, $\mathcal{K}$ field of fractions. Let $\mathcal{K}((\partial))$ be the skewfield of pseudodiffderential operators, $\mathcal{K}(\partial)$ the sub-skewfield of rational ones (i.e. the sub-skewfield, generated by $\mathcal{K}[\partial]$). Then

Theorem. (a) Any $H \in \text{Mat}\,_n(\mathcal{K}(\partial))$ can be represented as AB^{-1}, where $A, B \in \text{Mat}\,_n\mathcal{K}[\partial]$, B non-degenerate.

(b) There exists a *minimal* such representation $A_0B_0^{-1}$ so that any other is $(A_0C)(B_0C)^{-1}$, C non-degenerate.

(c) AB^{-1} is minimal iff $\text{Ker}\ A \cap \text{Ker}\ B = 0$ in any differential field extention of $\mathcal{K}$.

The best *proof.* Use the theory of non-commutative *principal ideal rings*.

What is a Hamiltonian equation

$$\frac{du}{dt} = H(\partial)\frac{\delta}{\delta u}\int h \tag{10}$$

when H is a non-local Poisson structure?

Fix a fractional decomposition $H = AB^{-1}$. We write association relation:

$$V/\partial V \ni \int h \overset{H}{\leftrightarrow} P \in V^{\ell}$$

if $P = A(\partial)F$, $\frac{\delta}{\delta u}\int h = B(\partial)F$ for some $F \in \mathcal{K}^{\ell}$. Then the equation (10) is interpreted as

$$\frac{du}{dt} = P \quad \left(\approx A(\partial)B(\partial)^{-1}\frac{\delta}{\delta u}\int h\right).$$

Lenard–Magri scheme for the (non-local) bi-Poisson structure (H, K) i.e. both H, K are Poisson and also $H + K$ is Poisson (all above examples are such). A bi-Hamiltonian equation:

$$\frac{du}{dt} = \underbrace{H(\partial)\frac{\delta}{\delta u}\int h_0 = K(\partial)\frac{\delta}{\delta u}\int h_1}_{\text{means}} := P_1 \tag{11}$$

$$\int h_0 \overset{H}{\leftrightarrow} P_1 \overset{K}{\leftrightarrow} \int h_1 \, .$$

Then under certain conditions the Hamiltonian equation (11) is integrable:

Theorem. Let $H = AB^{-1}$, $K = CD^{-1}$ be skewadjoint. Let $\{\xi_n\}_{n=-1}^N$, $\{P_n\}_{n=0}^N$ be sequences such that

$$(*) \qquad \xi_{n-1} \overset{H}{\leftrightarrow} P_n \overset{H}{\leftrightarrow} \xi_n \,, \quad n = 0, \ldots, N \,.$$

Then

(a) $(P_n|\xi_m) = 0$, $m \geq -1, n \geq 0$ (i.e. the $\int h_m$ are in involution if $\xi_m = \int h_m$ are exact)

(b) Provided that $H = AB^{-1}, K = CD^{-1}$ is a bi-Poisson structure, K non-degenerate, and ξ_{-1}, ξ_0 closed, we have: ξ_n are closed, hence exact in some differential algebra extention of V, and

$$[P_m, P_n] \subset \text{Ker } B^* \cap \text{Ker } D^* \,, \quad m, n \geq 0 \,.$$

(c) If the orthogonality conditions hold:

$$(\text{span } \{\xi_m\}_{m=-1}^N)^\perp \subset \text{Im } C$$
$$(\text{span } \{P_n\}_{m=0}^N)^\perp \subset \text{Im } B \,,$$

we can extend $(*)$ to infinity.

(d) If also ord $P_n \to \infty$, then each of the equations $\frac{du}{dt_n} = P_n$ is integrable and has infinitely many linearly independent integrals of motion in involution.

Classical Hamiltonian reduction for PVA V.

Let $\mu : R \to V$ be a Lie conformal algebra homomorphism; it extends to the PVA homomorphism $\mu : S(R) \to V$. Let $I_0 \subset S(R)$ be a PVA ideal. Let $I = V\mu(I_0)$ be the differthential algebra ideal of V, generated by $\mu(I_0)$. The classical Hamiltonian reduction is the differential algebra

$$\mathcal{W}(V, R, I_0) = (V/I)^{\mu(R)}$$

with the λ-bracket

$$\{f + I_\lambda g + I\} = \{f_\lambda g\} + I[\lambda].$$

Examples. Classical W-algebra, associated to ($\mathfrak{g}$, nilpotent f),

$\mathcal{W}(\mathfrak{g}, f)$ is obtained by taking

$V = S(\mathbb{F}[\partial]\mathfrak{g})$ with $[a_\lambda b] = [a, b] + (a|b)\lambda$,

$R = \mathbb{F}[\partial]\mathfrak{g}_{>0}, [a_\lambda b] = [a, b]$,

I_0 ideal of $S(R)$, generated by $m - (f|m)$, where $m \in \mathfrak{g}_{\geq 1}$.

Drinfeld–Sokolov, using f = principal nilpotent, constructed the integrable DS hierarchy. One can construct the generalized DS hierarchies for any nilpotent f, such that $f + s$ is a semisimple element of $\mathfrak{g}$, where s has maximal (ad h)-eigenvalue, using the language of PVA.

Dirac reduction for PVA.

Let V be a non-local PVA, let $\theta_1, \dots, \theta_m \in V$ (constraints), let I be the differential ideal of V generated by them. Consider the rational pseudodifferential operator $C(\partial)$ with symbol $(C_{\alpha\beta}(\lambda)) = (\{\theta_{\beta\lambda}\theta_\alpha)\})$.

Theorem. Assume that $C(\partial)$ is an invertible matrix pseudodifferential operator. Then

(a) $\{f_\lambda g\}^D := \{f_\lambda g\} - \sum_{\alpha,\beta=1}^m \{\theta_{\alpha\ \lambda+\partial} g\}_\to (C^{-1})_{\alpha\beta}(\partial+\lambda)\{f_\lambda \theta_\beta\}$ is again a (non-local) PVA structure on V.

(b) θ_i are central: $\{\theta_{i\ \lambda} f\}^D = 0$.

(c)V/I with the induced λ-bracket is again a (non-local) PVA.

Corollary. If $H = \begin{array}{c} m \\ n \end{array}\begin{pmatrix} A & B \\ -B^* & D \end{pmatrix}$ is a (non-local) Poisson structure in $m+n$ variables, then $A + BC^{-1}B^*$ is a non-local Poisson structure in m variables.

HAPPY BIRTHDAY

!! ROBERTO !!