Conformal Field Theory, Operator Algebras and Tensor Categories

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Operator algebraic approach to conformal field theory

$\rightarrow$ Interactions among subfactor theory, noncommutative geometry, vertex operator algebras and tensor categories through (super)conformal field theory (mainly with S. Carpi, R. Hillier, R. Longo, N. Suthichitranont and F. Xu)

Outline of the talk:

1. The Virasoro algebra and local conformal nets
2. Analogy between local conformal nets and differential geometry
3. The Dirac operator and supersymmetry
4. $N = 2$ supersymmetry, the character formulas, the subfactor theory and the noncommutative geometry
5. Moonshine, Virasoro frames and binary codes
6. Boundary conformal field theory

Our spacetime is now S^1 and the spacetime symmetry group is the infinite dimensional Lie group $\mathrm{Diff}(S^1)$. It gives the Lie algebra generated by $L_n = -z^{n+1}\dfrac{\partial}{\partial z}$ with $|z| = 1$, $z \in \mathbb{C}$.

The Virasoro algebra is a central extension of its complexification. It is the infinite dimensional Lie algebra generated by $\{L_n \mid n \in \mathbb{Z}\}$ and a central element c with the following relations.

$$[L_m, L_n] \ = \ (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}c.$$

We have a good understanding of its irreducible unitary highest weight representations, where the central charge c is mapped to a positive scalar. (This value is also called the central charge.)

Fix a nice representation π of the Virasoro algebra, called the vacuum representation, and simply write L_n for $\pi(L_n)$.

Consider $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$, called the stress-energy tensor, for $z \in \mathbb{C}$ with $|z| = 1$. Regard it as a Fourier expansion of an operator-valued distribution on S^1. This is a typical example of a quantum field.

Fix an interval (an open arc) I and take a C^∞-function f with $\operatorname{supp} f \subset I$. We have an (unbounded) operator $\langle L, f \rangle$ as an application of an operator-valued distribution to a test function.

Let $A(I)$ be the von Neumann algebra of bounded linear operators generated by these operators with various test functions. The family $\{A(I)\}$ gives one realization of one chiral conformal field theory.

Operator algebraic axioms: (chiral conformal field theory)

Motivation: Operator-valued distributions $\{T\}$ on S^1.

Fix an interval $I \subset S^1$, consider $\langle T, f \rangle$ with $\mathrm{supp}\ f \subset I$.

$A(I)$: the von Neumann algebra generated by these observables.

1. $I_1 \subset I_2 \Rightarrow A(I_1) \subset A(I_2)$.
2. $I_1 \cap I_2 = \varnothing \Rightarrow [A(I_1), A(I_2)] = 0$. (locality)
3. $\mathrm{Diff}(S^1)$-covariance (conformal covariance)
4. Positive energy
5. Vacuum vector

Such a family $\{A(I)\}$ is called a local conformal net. Its representation on another Hilbert space gives a subfactor of $A(I)$ through the Doplicher-Haag-Roberts endomorphism.

Geometric aspects of local conformal nets

Consider the Laplacian Δ on an n-dimensional compact oriented Riemannian manifold. Recall the classical Weyl formula:

$$\mathrm{Tr}(e^{-t\Delta}) \sim \frac{1}{(4\pi t)^{n/2}}(a_0 + a_1 t + \cdots),$$

where the coefficients have a geometric meaning.

The conformal Hamiltonian L_0 of a local conformal net is the generator of the rotation group of S^1. For a nice local conformal net, we have an expansion

$$\log \mathrm{Tr}(e^{-tL_0}) \sim \frac{1}{t}(a_0 + a_1 t + \cdots),$$

where a_0, a_1, a_2 are identified. (K-Longo) This gives an analogy between the Laplacian Δ and the conformal Hamiltonian L_0. The "square root" of the former is the classical Dirac operator.

Noncommutative geometry:

Noncommutative operator algebras are regarded as function algebras on noncommutative spaces.

In geometry, we need manifolds rather than compact Hausdorff spaces or measure spaces.

The Connes axiomatization of a noncommutative compact Riemannian spin manifold: a spectral triple $(\mathcal{A}, H, D)$.

1. $\mathcal{A}$: $*$-subalgebra of $B(H)$, the smooth algebra $C^\infty(M)$.
2. H: a Hilbert space, the space of L^2-spinors.
3. D: an (unbounded) self-adjoint operator with compact resolvents, the Dirac operator.
4. We require $[D, x] \in B(H)$ for all $x \in \mathcal{A}$.

$N = 1$ super Virasoro algebras: (Adding a square root of L_0)

The infinite dimensional super Lie algebras given by more generators and relations. One new relation is $G_0^2 = L_0 - c/24$, so this gives a square root of L_0 which is analogous to the Laplacian.

We again consider a unitary representation of (one of) the $N = 1$ super Virasoro algebras. Consider operator-valued distributions and test functions as before, we obtain a family $\{A(I)\}$ of von Neumann algebras parameterized by $I \subset S^1$. This gives a superconformal net, for which now the bracket $[x, y]$ means a graded commutator.

To make an interesting study in connection to noncommutative geometry, we need $N = 2$ super Virasoro algebra and its unitary representations, where we add two series of new generators. (Carpi's talk on Monday.)

It has been known an irreducible unitary representation of the $N = 2$ super Virasoro algebra maps c to a scalar in the set $\{3m/(m+2) \mid m = 1, 2, 3, \dots\} \cup [3, \infty)$. We consider only the case $c = 3m/(m+2)$ in the discrete series now.

We fix the vacuum representations, use the four operator-valued distributions arising as before, and obtain a family of von Neumann algebras $\{A(I)\}$.

We use the coset construction, arising from theory of infinite dimensional Lie algebras. However, it is unclear whether the representation of the $N = 2$ super Virasoro algebra contains all of the coset or not. This equality is often taken as a "theorem", but we have been unable to find a complete proof in literature.

This problem is directly related to the one for the explicit character formulas, and the operator algebraic methods give a proof of this equality for the coset and the character formula.

The $N = 2$ superconformal nets are the extensions of these coset nets by definition. They are classified and listed completely.

We now connect these to noncommutative geometry by constructing a family of spectral triples parameterized by the intervals I. We need the Dirac operator, and just choose G_0^1, and put $\delta(x) = [G_0^1, x]$ for a bounded linear operator x on the representation space.

We put $\mathcal{A}(I) = A(I) \cap \bigcap_{n=1}^{\infty} \mathrm{dom}(\delta^n)$, and prove that each $\mathcal{A}(I)$ is strongly dense in $A(I)$ and satisfies $\delta(\mathcal{A}(I)) \subset \mathcal{A}(I)$.

We study entire cyclic cohomology introduced by Connes, a nice cohomology theory for an infinite dimensional noncommutative manifold. Our Dirac operator satisfies the θ-summability condition.

A JLO cocycle for such a spectral triple, an element in the entire cyclic cohomology, is defined. We have the index pairing between the K_0-group and the entire cyclic cohomology, producing a number.

We consider the spectral triples arising from certain Ramond representations, which produce subfactors and then the K_0-elements of a certain $*$-subalgebra, and each gives a different JLO-cocycle. Our result then says that the pairing gives the Kronecker δ.
(Carpi-Hillier-K-Longo-Xu — Carpi's talk on Monday).
(Fröhlich-Gawędzki suggested connections of noncommutative geometry and superconformal field theory.)

Moonshine

Mysterious relations between the Monster group, the sporadic finite simple group having the largest order, and elliptic modular functions such as the j-function.

Formulated first by Conway-Norton based on McKay's observation and the realization in terms of vertex operator algebra (VOA), which is an algebraic axiomatization of Wightman fields on S^1, has been given by Frenkel-Lepowsky-Meurman. It is called the Moonshine VOA, and the full conjecture has been solved by Borcherds.

A local conformal net and a vertex operator algebra should describe the same physical theory based on different mathematics. So we expect deep relations between the two mathematical theories.

The operator algebraic counterpart has been constructed by K-Longo.

One of the most fundamental VOA is a Virasoro VOA with $c = 1/2$, written as $L(1/2, 0)$. This corresponds to the Ising model.

It has been recognized that the Moonshine VOA is an extension of the 48th tensor power of the Virasoro VOA with $c = 1/2$. This has some interpretation from a viewpoint of lattice theory, and based on that, a tensor power of $L(1/2, 0)$ is called a Virasoro frame in general, and its extension is called a framed VOA.

We now recall a theory of binary code. It is simply a subspace of a vector space $\mathbb{F}_2^k$ over the field $\mathbb{F}_2$ of order 2. It is an extremely easy group embedded into another extremely easy group, but the way of embedding can be highly nontrivial, and this situation is somehow formally similar to subfactor theory.

An element of a binary code is called a code word. The number of 1 in a code word is called weight. If the weight of any code word is a multiple of 2, 4 and 8, then the binary code is said to be even, doubly even, triply even, respectively.

The Golay code is a famous exceptional code of length 24. This is related to the Leech lattice, which is the exceptional lattice among 24 even self-dual lattices in dimension 24, and this is also realizes the densest lattice sphere packing in dimension 24.

Lam-Yamauchi recently established a systematic construction of a family of framed VOA's from a triply even binary code D of length being a multiple of 16 containing the vector $(1, 1, \ldots, 1)$. The Moonshine VOA is such an example.

We have a notion of the dual code $C = D^{\perp}$, and Lam-Yamauchi produced a VOA having the structure code (C, D) from a given code D satisfying the above conditions.

The corresponding operator algebraic construction has been found by K-Suthichitranont recently. The Lam-Yamauchi method uses many techniques from theory of binary codes and it looks difficult to translate their method directly. We have thus taken an entirely different method.

Two big advantages of the operator algebraic methods are Jones index and α-induction. We construct the local conformal net as a "two-step crossed product". (It was first proved that the Moonshine net is of this form by us and later they found the corresponding result in the VOA theory.)

Boundary conformal field theory:

We now study a family of operator algebras parameterized by certain rectangles in the $(1+1)$-dimensional half Minkowski space $\{(x,t) \mid x > 0\}$. This is our framework of boundary conformal field theory due to Longo-Rehren. They have shown some holographic correspondence principle under the assumption of the Haag duality.

It reduces some study on the half space to that on the boundary. Based on this correspondence, we have a classification result for the central charge $c < 1$ (K-Longo-Pennig-Rehren).

In the case of the entire $(1+1)$-dimensional Minkowski space, we have a similar formulation and it is called a full conformal field theory. The relation between the two theories have been also studied by Longo-Rehren by shifting the boundary to infinity.

It was a procedure to remove the boundary. We now consider the converse problem. That is, starting from a full conformal field theory, we would like to add the boundary without affecting the local structure of the theory.

That is, for rectangles in the half space away from the boundary, we would like to keep the original operator algebras, and create a consistent behavior near/at the boundary.

This has been done by Carpi-K-Longo recently with purely operator algebraic technique. A classification problem for boundary conformal field theory arising in this way seems to be feasible and we have some advances, since we have shown this is a finite problem and our previous classification machinery should be useful.

This classification problem is related to the one studied in the context of topological field theory by Fuchs, Kong, Runkel and Schweigert, in terms of tensor categories. We believe their algebraic approach and our operator algebraic approach are essentially the same and expect more (noncommutative) geometric interpretations.

Another related problem is a deformation procedure due to Longo-Witten. Though their method does not involve conformal symmetry and has only less symmetry, it provides a method to produce new examples. Such constructions have been studied by Bischoff and Tanimoto. (Tanimoto's talk on Monday.)

This construction arose from Witten's physical intuition and has some geometric background, so some new connections of these approaches also look quite interesting.

Buon Compleanno, Roberto!