Connected components of compact matrix quantum groups

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Introduction

Quantum groups originate in the theory of Hopf algebras, which in turn has its roots in

1) *algebraic topology* (Hopf, '40, Borel '50),
2) *algebraic groups* (Dieudonné, Cartier, '50)
3) *duality for locally compact groups* (G.I. Kac '60, Takesaki '70)

•The first example, due to Hopf, was the *cohomology ring* H of a Lie group (or more general manifolds with a non-associative product operation),

$$G \times G \to G$$

inducing the *coproduct*

$$\Delta : H \to H \otimes H.$$

The term *Hopf algebra* was conied by Borel ('53), as an abstraction of H. The original axioms assumed H to be graded, graded–commutative... Structure theorems were obtained.

Cartier ['55] removed many of the original restrictions. His definition in modern terms is quite close to the notion of a *cocommutative filtered Hopf algebra*

$$\Sigma\Delta = \Delta, \qquad \Sigma h \otimes h' = h' \otimes h$$

Main known examples of this early period were

- the *cocommutative universal enveloping algebra* of a classical Lie group,

$$\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$$

$$x \in \mathfrak{g} \mapsto x \otimes 1 + 1 \otimes x$$

- the *commutative algebra* of representative functions on a compact Lie group G,

$$\Delta : f(g) \in \mathcal{R}(G) \to f(gh) \in \mathcal{R}(G) \otimes \mathcal{R}(G).$$

- the Hopf-von Neumann algebras $L^\infty(G)$, $L(G)$ where G is a locally compact group.

- Until the mid 80s, few examples were known which were not either commutative or cocommutative. These were discovered with the advent of *quantum groups*, by Drinfeld and Jimbo as deformations of the classical groups,

$$U_q(\mathfrak{g}).$$

• Woronowicz (1987) initiated an operator algebraic approach, motivated by Connes noncommutative geometry, and gave an abstract definition of *compact matrix quantum group*, later generalized to *compact quantum group*.

A CMQG is an abstract Hopf C^*–algebra generated by the coefficients of a defining representation

$$G = (A_G, \Delta, u), \quad u \in M_n(A_G)$$

Examples of CMQG are:

• compact Lie groups,

$$A_G = C(G)$$

all the commutative examples

• $\mathrm{SU}_q(d)$, G_q (duals of $U_q(\mathfrak{g})$, $q > 0$)

• finitely generated *discrete* groups

$$A_G = C^*(\Gamma),$$

'all' the cocommutative examples.

Irreducible reps are 1-dimensional, $\widehat{G} = \Gamma$, analogue of abelian groups

• $A_o(F)$, $A_u(F)$
free analogue of orthogonal and unitary groups, Wang, Van Daele

More important examples exist which I have not mentioned.

Woronowicz proved
• Haar measure,
• Peter-Weyl theory
• dense Hopf *–subalgebra of 'representative functions'

• CQG are approximated by CMQG

• Tannaka-Krein duality: A CQG is roughly the same as a tensor C^*–category together with an embedding

$$H : \mathcal{C} \to \text{Hilb}.$$

The correspondence is given by

$$\mathcal{C} = \text{Rep}(G).$$

Main new constructions

• Free products: $G * G'$ (Wang)

• Unlike classical compact Lie groups, classification of *all* CMQG is intractable.

• An active field is classification of CMQG with representation ring isomorphic to that of a given Lie group. Or of quantum groups with isomorphic representation categories.

For example, this is solved for SU(2): (Banica ’97)

$$R(A_o(F)) = R(\mathsf{SU}(2))$$

$$\mathsf{Rep}(A_o(F)) = \mathsf{Rep}(\mathsf{SU}_q(2)), \text{ suitable } F, q,$$

But in general it is a difficult problem. CMQG are very many, may be highly noncommutative.

• We are interested in studying the general structure. *To what extent can CMQG be considered as generalizations of Lie groups?*

If no restriction on the class is made, analogy with Lie groups is rather weak. All f.g. groups discrete are included!

Although CQG do not fit precisely the needs of algebraic low dim QFT (Szlachanyi’s WHA would be more appropriate), original interest in this project was in those with *commutative fusion rules*.

The problem involves a unification of the theory of compact Lie groups with certain aspects of geometric group theory. For this reason, it turns out useful to describe CQG as *discrete* mathematical objects, passing to the dual. Namely, as tensor C^*–categories.

For compact Lie groups, *connectedness* is a basic property. Not only this, but *local connectedness* enters, in a crucial way, together with *finite dimensionality*, (we do not consider either of them, here) to characterize Lie groups among the locally compact ones, by the solution to Hilbert fifth problem of Gleason, Montgomery and Zippin (50s).

We aim to

• Introduce the notion of *identity component* G_0 of a compact quantum group which

-extends the classical notion for compact groups and

-reduces to connectedness in the sense of Shuzhou Wang if $G = G_0$.

• consider the noncommutative analogue of the following facts for compact Lie groups:

G_0 *is a normal subgroup.*

G/G_0 *is a finite group.*

Normal quantum subgroups

Subgroups are described by *epimorphisms* of Hopf C^*–algebras, the *'restriction map'* (Podles)

$$\mathcal{A}_G \twoheadrightarrow \mathcal{A}_K$$

Consider the right translation of G by K,

$$\rho : A_G \to A_G \otimes A_K,$$

as well as the left translation,

$$\lambda : A_G \to A_K \otimes A_G$$

We may thus consider the analogue of the right and left K–invariant functions,

$$A_{G/K} := \{a \in A_G : \rho(a) = a \otimes 1\},$$

$$A_{K\backslash G} := \{a \in A_G : \lambda(a) = 1 \otimes a\},$$

and also the analogue of the bi–K–invariant functions:

$$A_{K\backslash G/K} := A_{K\backslash G} \cap A_{G/K}.$$

$A_{G/K}$ and $A_{K\backslash G}$ are globally G–invariant,

$\Delta(A_{K\backslash G}) \subset A_{K\backslash G} \otimes A_G, \qquad \Delta(A_{G/K}) \subset A_G \otimes A_{G/K}.$

It follows that

$$\Delta(A_{K\backslash G/K}) \subset A_{K\backslash G} \otimes A_{G/K}.$$

Definition *(Wang) A subgroup N of G is normal if it satisfies the following equivalent properties,*

a) $A_{N\backslash G} = A_{G/N}$,

b) $\Delta(A_{G/N}) \subset A_{G/N} \otimes A_{G/N}$.

c) For any $v \in \widehat{G}$ such that $v \restriction_N > \iota$, then

$$v \restriction_N = \mathit{dim}(v)\iota$$

Equivalence follows from the fact that $A_{N\backslash G}$ is generated by coefficients $v_{\psi,\phi}$ with ψ $N-$invariant and ϕ arbitrary, while for $A_{G/N}$ we need ψ arbitrary and ϕ $N-$invariant.

Hence if N is normal, $A_{G/N}$ becomes a compact quantum group with the restriction of the coproduct of G.

By a), this notion reduces to the classical notion of normality.

The definition does not mention the adjoint action, but it is equivalent (Wang).

Example If $G = C^*(\Gamma)$, any quantum subgroup K of G is normal,

$$K = C^*(\Gamma/\Lambda), \quad G/K = C^*(\Lambda),$$

with $\Lambda \triangleleft \Gamma$.

Connected compact quantum groups

Definition (Wang, 2002) *A compact quantum group G is connected if A_G admits no non trivial finite dimensional unital Hopf $*-$ subalgebra.*

In the classical case this definition says that the only finite group Γ for which there is a continuous epimorphism

$$G \to \Gamma$$

is the trivial group. This is obviously weaker than connectedness, but it is in fact equivalent since if G is disconnected, we have

$$G \to G/G_0$$

and G/G_0 is totally disconnected, hence it has non trivial finite quotients.

Definition A representation u of a cqg G will be called a *torsion* representation if the subhypergroup

$$< u, \overline{u} > \subset \widehat{G}$$

is finite.

Proposition G is connected if and only if it admits no non trivial (irreducible) torsion representations.

In particular, quantum groups with fusion rules identical (or quasiequivalent) to those of connected compact groups are connected.

Examples

Most known examples are connected:

• If G is a classical compact Lie group, G_q is connected.

• products of connected cqg are connected.

• quotient qg, i.e. Hopf C^*–subalgebras,

$$\mathcal{A}_L \hookrightarrow \mathcal{A}_G$$

of connected qg are connected.

• If $N \triangleleft G$ and G/N are connected then G is connected

• $A_u(F)$ and $A_o(F)$ are connected.

• If $G = C^*(\Gamma)$, with Γ a discrete group, the irreducibles of G are the elements of Γ, hence G is connected if and only if Γ is torsion-free.

• $U_q(su(2))$, $0 < q < 1$,

$$KEK^{-1} = qE, \qquad KFK^{-1} = q^{-1}F,$$

$$[E, F] = \frac{K^2 - K^{-2}}{q - q^{-1}},$$

$$E^* = F, \qquad K^* = K.$$

There are four 1–dimensional representations,

$$\varepsilon_\omega : E \to 0, \quad F \to 0, \quad K \to \omega \in \mathbb{Z}_4,$$

Only two are *–representations, $\varepsilon_{\pm 1}$.

G is not connected as ε_{-1} is torsion of order 2.

All the *–irreps are of the form

$$\varepsilon_\pm \otimes \pi_n = \pi_n \otimes \varepsilon_\pm.$$

π_n of dim $n+1$ with positive weights

$$G = \mathrm{SU}_q(2) \times \mathbb{Z}_2 \quad \text{(Rosso)}$$

The identity component of a CQG

Classical case

For locally compact compact groups G, duality theorems allow to *determine* the identity component from the *dual object* $\widehat{G}$. Hence, in the compact case, in algebraic terms.

Let $\widehat{G}$ be the dual hypergroup (set of irreps with $\otimes$ and conjugation) and

$\widehat{G}^{\mathrm{tor}} = \{u \in \widehat{G}$ generating a finite subhypergp$\}$.

Then (Pontryagin, Iltis):

$$\widehat{G/G_0} = \widehat{G}^{\mathrm{tor}},$$

$$G_0 = \{g \in G : u(g) = 1, u \in \widehat{G}^{\mathrm{tor}}\}.$$

Hence G_0 corresponds to the process of *eliminating torsion in* $\widehat{G}$.

G is totally disconnected iff $\widehat{G} = \widehat{G}^{\mathrm{tor}}$.

• In the general case, these ideas do *not* suffice to *define* G_0, since different quantum groups, may have the *same* hypergroup.

Unlike the classical groups, this may happen even among the connected ones! (e.g. $A_o(F)$ and SU(2))

To define G_0 we use instead the representation category

$$\widehat{G} \quad \textit{vs} \quad \mathrm{Rep}(G)$$

Quantum case

Definition G_0 is the quantum subgroup 'generated' by all the connected quantum subgroups K of G. In other words,

$$(u,v)_{G_0} = \cap_K (u \restriction_K, v \restriction_K).$$

Proposition *G_0 is the largest connected quantum subgroup of G*

Corollary *Every torsion representation of G restricts to the trivial representation of G_0. This reads, if G_0 is normal,*

$$\mathit{Rep}(G)^{\mathit{tor}} \subset \mathit{Rep}(G/G_0).$$

In the classical case, the converse holds as well by profiniteness of G/G_0, but it does not hold for CQG.

Example If $G = C^*(\Gamma)$, then

$$A_{G_0} = C^*(\Gamma/\rho(\Gamma)), \quad A_{G/G_0} = C^*(\rho(\Gamma))$$

where $\rho(\Gamma)$ is the *torsion-free radical* of G, of Brodsky and Howie, i.e. the unique minimal normal subgroup such that $\Gamma/\rho(\Gamma)$ is torsion-free.

• The results of B-H when interpreted for quantum groups mean that under certain conditions

$$G_0 \text{ contains a 1-dim torus}$$

If Γ^{tor} is a subgroup,

$$\rho(\Gamma) = \Gamma^{\mathrm{tor}}.$$

In general, only $\rho(\Gamma) \supset \mathrm{Normal}(\Gamma^{\mathrm{tor}})$ holds.

Example (Chiodo and Vyas, 2011)

$$\Gamma = (\mathbb{Z}_m * \mathbb{Z}_n) *_{xy=z^p} \mathbb{Z},$$

$$\Gamma^{\text{tor}} = \{\text{conjugates to } x \text{ or } y\},$$

$$\Gamma/\text{Normal}(\Gamma^{\text{tor}}) = \mathbb{Z}_p$$

- Computing $\rho(\Gamma)$:

Set

$$N_1 = \text{Normal}(\Gamma^{\text{tor}}),$$

$$N_r = \text{Normal}(\gamma \in \Gamma : \gamma^n \in N_{r-1}).$$

Then

$$N_1 \subset N_2 \subset \dots$$

$$\rho(\Gamma) = \cup_r N_r.$$

Main results

Examples with non-normal G_0

Theorem *Let G_c be a connected compact quantum group and let Γ be a discrete group and consider the free product quantum group*

$$G = G_c * C^*(\Gamma).$$

a) If G_c has a non-trivial irreducible representation of dimension > 1 and Γ has a non- trivial element of finite order then G_0 is not normal.
b) If $\Gamma = \Gamma^{tor}$ then $G_0 = G_c$,
c) If G_c is a semisimple Lie group, G has no no-trivial normal connected subgroup.

Being free products of quantum groups, these examples are highly noncommutative.

sketch of proof of a) The main ideas are that a normal subgroup $N \triangleleft G$ always corresponds to a full *normal tensor subcategory*

$$\mathrm{Rep}(G/N) \triangleleft \mathrm{Rep}(G).$$

and that we know all the irreps of a free product quantum group.

If G_0 were normal, the irreps structure of free products allows to find one,

$$\overline{u}\gamma u \in \mathrm{Rep}(G/G_0), \quad u \in \widehat{G_c},\ \gamma \in \Gamma,$$

with non trivial restriction to G_c if $\dim(u) > 1$ and $\gamma \in \Gamma^{\mathrm{tor}}$. But γ becomes trivial on G_0, hence $\overline{u}\gamma u$ becomes $\overline{u}u$ on G_0, which is not trivial.

Normal tensor subcategories

• Normality of a full tensor subcategory of $\mathrm{Rep}(G)$ is a condition that generalizes the notion of normal subgroup of a discrete group:

If $\Lambda \subset \Gamma$ is an inclusion of groups,

$$L = C^*(\Lambda), \quad G = C^*(\Gamma),$$

$\mathrm{Rep}(L)$ is normal in $\mathrm{Rep}(G)$ iff Λ is normal subgroup in Γ.

• In the classical case every full tensor subcategory of Rep(G) is normal.

• In general, normality of

$$\mathrm{Rep}(L) \subset \mathrm{Rep}(G)$$

is a special case of the condition that characterizes homogeneous spaces of G

$$A_L \to A_G$$

arising from quantum subgroups, $L = G/K$. (P-Roberts '06). The speciality corresponds to the fact that the subgroup is normal (i.e. L is a quantum group)

$$L = G/N.$$

Normality of a tensor subcategory is defined by the following equivalent conditions.

Theorem *Let $\mathcal{S} \subset$ Rep(G) be a full tensor C^*–category with conjugates and $\mathcal{A}_L$ the corresponding quotient qg. The following conditions are equivalent. For any irreducible $u \in \mathcal{S}$,*

a) $1_{\overline{u}} \otimes H_v \otimes 1_u \circ R \subset H_{(\overline{u}vu)_{\mathcal{S}}}, \quad R \in (\iota, \overline{u}u), v \in \mathcal{S}$ *irreducible,*

b) $\sum_i u_{ij}^* x u_{i,s} \in \mathcal{A}_L, \quad x \in \mathcal{A}_L$,

c) *there is a normal compact quantum subgroup N such that $L = N \backslash G$*

Totally disconnected CQG

Definition *If G_0 is the trivial group, G will be called totally disconnected*.

Definition *A compact quantum group will be called profinite if its Hopf C^*-algebra is the inductive limit of finite dimensional Hopf C^*-subalgebras.*

- Profiniteness implies total disconnectedness

- A CQG is profinite iff *every* representation is torsion.

- If every *irreducible* representation of Rep(G) is a torsion object, then G is totally disconnected.

Indeed, irreps of G_0 are restrictions of irreps of G. They need to be torsion, hence trivial.

• For example, if Γ is a *torsion group*,

$$\Gamma = \Gamma^{\text{tor}},$$

then $G = C^*(\Gamma)$ is *totally disconnected*,

$$G_0 = \{1\}.$$

In this case, the finiteness problem for G/G_0 becomes the question: *Is any finitely generated torsion group finite?*

This is precisely the Burnside problem, which has a negative answer.

Proposition *Let Γ be a finitely generated, infinite torsion group, with generators $g_1, \dots g_n$. Then*

$$u = g_1 \oplus \cdots \oplus g_n$$

is a non torsion representation of $C^*(\Gamma)$. Hence $C^*(\Gamma)$ is not finite but totally disconnected.

• Literature on the various problems of Burnside, Milnor and von Neumann for discrete groups provides many non-amenable examples (Golod-Shafarevich '64, Olshanskii '80, Adian '83, Ershov 2011), but also of intermediate growth, hence amenable (Grigorchuk '84).

• If $G_0 \neq \{1\}$ then G/G_0 may be infinite even if its representations, regarded as G–reps, are assumed to commute tensorially with every other representation of G! This is due to the fact that there are finitely generated (in fact f. presented) Γ with infinite Γ^{tor} and at the same time satisfy (Remeslennikov '74).

$$\Gamma^{\text{tor}} \subset Z(\Gamma).$$

• On the positive side, almost nilpotent discrete groups Γ have finite torsion subgroup Γ^{tor}. In addition, by Milnor, Wolf, Gromov theorems (late 60s-80s) they are precisely those with polynomial growth...

The torsion subcategory

$$\text{Rep}(G)^{\text{tor}} := \text{full}\{\text{torsion reps of } G\}$$

• If G_0 is normal and G/G_0 is profinite then

$$\text{Rep}(G)^{\text{tor}} = \text{Rep}(G_0 \backslash G)$$

hence,
- $\text{Rep}(G)^{\text{tor}}$ has $\otimes, \oplus$
- $\text{Rep}(G)^{\text{tor}}$ is a *normal* subcategory of $\text{Rep}(G)$.

But in general, $\text{Rep}(G)^{\text{tor}}$ may behave badly.

• $\text{Rep}(G)^{\text{tor}}$ has conjugates and subobjects.

• For a discrete group, $G = C^*(\Gamma)$,

$$\{\text{irreps of Rep}(G)^{\text{tor}}\} = \Gamma^{\text{tor}},$$

hence $\text{Rep}(G)^{\text{tor}}$ may easily be not tensorial when noncommutative.

• Rep$(G)^{\mathsf{tor}}$ may lack direct sums, even if it has tensor products (e.g. all infinite f.g. torsion groups).

In general, torsion gives little information onf G_0, since,

$< \mathrm{Rep}(G)^{\mathsf{tor}}, \otimes, \oplus > \subset \{u \in \mathrm{Rep}(G) : u \restriction_{G_0} = 1\}$.

Moreover, the inclusion may be strict (CV examples)

• The following inclusions are strict for CMQG

$$\{\text{tot disc}\} \supset \{\text{torsion irreps}\} \supset \{\text{finite}\}$$

For the first, the reason is that CV examples are totally disconnected. Indeed, for all $\gamma \in \Gamma$, $\gamma^p \in \mathrm{Normal}(\Gamma^{\mathsf{tor}}) \subset \rho(\Gamma)$, hence $\gamma \in \rho(\Gamma)$ since $\Gamma/\rho(\Gamma)$ is torsion-free. Thus $\rho(\Gamma) = \Gamma$.

If G/G_0 is not profinite, one can derive information on *maximal normal connected subgroup* $G^{\mathsf{n}} \subset G_0$ from torsion.

Torsion degree

• Generalizing the computation of $\rho(\Gamma)$, we make an inductive process to eliminate torsion in $\text{Rep}(G)$. The result is a canonical sequence of normal subgroups,

$$G^0 = G \supset G^1 \supset \dots$$

Except for now it is not clear to us whether the limit of this sequence is connected. We thus use transfinite induction and extend this sequence to the ordinals, G_α, which must stabilize for cardinality reasons.

Definition torsion degree$(G) =$ smallest δ s.t

$$G_\delta = G_{\delta+1}.$$

It is an invariant measuring complexity of torsion.

Theorem *The torsion degree of G is the smallest ordinal δ such that G_δ is connected. Moreover,*

$G_\delta = G^n =:$ *maximal connected normal subgp.*

- If G_0 is normal and G/G_0 is profinite,

$$\text{torsion degree}(G) \in \{0, 1\},$$

with 0 corresponding to connected groups.

- For discrete groups, torsion degree $\leq \omega$.

- torsion degree(Chiodo – Vyas) $= 2$

- torsion degree(Burnside exs) $= 1$

- A generalization of examples due to Chiodo and Vyas, shows that all the ordinals $\leq \omega$ are realized by discrete groups.

Theorem *For the examples with non-normal G_0:*

$$G = G_c * \Gamma$$

with G_c semisimple Lie group and $\Gamma = \Gamma^t$, we have:
a) torsion degree(G) ≤ 2,
b) torsion degree(G) $= 1$ if $Z(G_c) = \{1\}$,
c) torsion degree(G) $= 2$ for $G_c = \mathrm{SU}(2)$.

a) is due to the fact that the subgroups turn out to be central in G_c.

Normality of G_0 and finiteness of G/G_0

Theorem *For a CQG G the following are equivalent,*
a) G_0 is normal and G/G_0 is finite,
b) Rep(G)^{tor} is tensorial, finite and normal.
In this case,
c) torsion degree(G) ≤ 1
d) Rep(G/G_0) $=$ Rep(G)^{tor}.

sketch of proof a)$\Rightarrow$ b) is easy. b)$\Rightarrow$ a) This includes the problem of showing that torsion degree(G) ≤ 1. There is $G_1 \triangleleft G$ such that Rep$(G)^{\text{tor}}$ = Rep(G/G_1). Since reps of Rep$(G)^{\text{tor}}$ are trivial on G_0 then $G_1 \supset G_0$. Hence the theorem amounts to show that G_1 is connected. Information from TK process shows that every torsion rep of G of restricts to trivial on G_1. Hence we need to show that free reps of G restrict to free reps on G_1. In the classical case this follows from a theorem of Clifford (later greatly developed by Mackey) on the analysis of restrictions of reps to a normal subgroup with finite index. In general, to understand restriction $u \restriction_{G_1}$ we make a detailed use of the theory of induction for tensor C^*–categories developed with Roberts in 2009.

Examples show that all the conditions are independently needed.

The Lie property

The previous characterization is a useful reduction of the problem.

• One one hand, the examples show that for positive results to our problem, we should take into account *commutativity* on $\widehat{G}$, or at least on $\widehat{G}^{\text{tor}}$. The latter will not suffice, by Remeslennikov examples. And perhaps neither the former.

• We look for *geometric conditions* on G.

On the other hand, finiteness of $\text{Rep}(G)^{\text{tor}}$ is a special case of a more fundamental problem, of interest independently of commutativity of $\widehat{G}$: We need at least to have it finitely generated!

• *For which cmqg G is any quotient quantum group*

$$A_L \hookrightarrow A_G$$

again matrix?

Definition *We call such cqg of Lie type. Equivalently,* $\widehat{G}$ *satisfies an ascending chain condition on subhypergroups: Every increasing sequence*

$$L_1 \subset L_2 \subset \dots$$

eventually stabilizes.

• In the classical case, quotients of compact Lie groups are Lie, hence of Lie type.

• For discrete groups, the corresponding property becomes *Noetherianity*: every subgroup is finitely generated.

• {f.g. almost nilpotent} $\subset$ {almost polycyclic}

$\subseteq$ {Noetherian group ring} $\subset$ {Noetherian group}

• $\subseteq$ is a long-standing open problem.

Examples

• CQG with representation ring isomorphic to that of a Lie group are of Lie type. For example, $A_o(F)$ and G_q are of Lie type.

• $A_u(F)$ is not of Lie type. This is the analogue of the fact that $\mathbb{F}_2$ is not Noetherian.

Theorem *If the representation ring*

$$R(G) := \mathbb{Z}\widehat{G}$$

is left Noetherian (a.c.c. on left ideals) then G is of Lie type.

Conclusions

Corollary *Let G be a CQG group of Lie type with commutative torsion subcategory Rep$(G)^{tor}$. Then Rep$(G)^{tor}$ is tensorial and finite. Hence, if also normal,*

- G_0 *is normal in* G
- G/G_0 *is finite,*
- *Rep*$(G/G_0) =$ *Rep*$(G)^{tor}$

Corollary *If $R(G)$ is commutative and finitely generated (as a ring!) then it is Noetherian, hence of Lie type.*

Problems

Can one generalize the previous corollary to just commutative fusion rules for CMQG?

Ring finite generation (of $R(G)$) implies hypergroup finite generation (of $\widehat{G}$). The converse holds in the classical case (Atiyah). We do not know whether it holds in general, assuming commutativity of $R(G)$.

We tend to believe answers are negative, for the following remarks:

• For compact connected Lie groups an abstract characterization of $R(G)$ is known (Osse, 1997). The simplest axiom is that $R(G)$ is finitely generated as a ring.

• The last corollary is a special case of a recent theorem of Hashimoto (2005) in geometric invariant theory. (However, our proof is independent.)

Hashimoto showed that if

$$A \subset B$$

are commutative algebras over some commutative Noetherian ring such that B is finitely generated and A is pure (e.g. a direct summand), then A is finitely generated.

This originates from the problem of finite generation of rings of polynomial invariants of algebraic groups acting on a polynomial ring. [Hilbert, Nagata, Mumford...]

G_0 for the compact real form of $U_{<0}(\mathfrak{sl}_2)$

Real forms:

$$E^* = F, \quad K^* = K, \qquad U_q(\mathfrak{su}_2),$$

$$E^* = -F, \quad K^* = K, \qquad U_q(\mathfrak{su}_{1,1}).$$

$U_q(\mathfrak{su}_2)$ has no f.d. *–representation on a Hilbert space.

$U_q(\mathfrak{su}_{1,1})$ admits two inequivalent irreducible Hilbert space *–reps for each dimension

$$u_{\pm n}$$

that can be explicited computed. They all commute.

$$\varepsilon_{-1} : K \to -1,\ E,\ F \to 0$$

is a nontrivial torsion *–representation.

$u_{\pm 1}$ satisfy

$$\overline{u_1} = u_{-1}, \quad u_1^2 > \varepsilon_{-1}$$

Fusion rules show that every irreducible admits a polynomial expression in ε_{-1} and u_1, hence $R(U_q(\mathfrak{su}_{1,1}))$ is a Noetherian commutative ring.

It can be checked explicitly that

$$\mathrm{Rep}(U_q(\mathfrak{su}_{1,1}))^{\mathrm{tor}} = \langle \varepsilon_{-1} \rangle$$

and that it is normal. Hence

$$G/G_0 = \mathbb{Z}_2.$$

Moreover,

$$G_0 = \mathrm{SU}_{|q|}(2).$$