The Kadison-Singer problem in II_1 factor framework

Rome, July 8th 2013

Sorin Popa

The work of Kadison-Singer in 1959

The KS pure state extension problem

Given a maximal abelian *-subalgebra (MASA) $A \subset \mathcal{B}(\ell^2\mathbb{Z})$ (notably, the diagonal MASA, $A = \ell^\infty\mathbb{Z}$, or the diffuse MASA, $A = L(\mathbb{Z})$), does any pure state on A extend to a unique state on $\mathcal{M} = \mathcal{B}(\ell^2\mathbb{Z})$? More generally, we'll consider this question/property for MASAs in arbitrary vN algebras, $A \subset \mathcal{M}$, and call it the *KS problem/property for* $A \subset \mathcal{M}$.

Theorem (Paving reformulation of KS problem)

Let $A \subset \mathcal{M} = \mathcal{B}(\ell^2\mathbb{Z})$ be a MASA (more generally A a MASA in an arbitrary vN algebra $\mathcal{M}$). Then $A \subset \mathcal{M}$ has the KS property iff it satisfies

The paving property: $\forall x \in \mathcal{M}$, $\forall \varepsilon > 0$, $\exists q_k \in \mathcal{P}(A)$ a finite partition of 1 such that $d(\Sigma_k q_k x q_k, A) \leq \varepsilon d(x, A)$ (any $x \in \mathcal{M}$ can be ε-*paved*, $\forall \varepsilon > 0$).

Moreover, if these conditions are satisfied then: there exists a unique conditional expectation E of $\mathcal{M}$ onto A, it is unique, and $\lim_n \Sigma_k p_k^n x p_k^n = E(x)$, $\forall \varepsilon_n$-paving $\{p_k^n\}_k$, with $\varepsilon_n \to 0$; also, $\forall$ pure state ψ on A, $\psi \circ E$ is the unique state extension of ψ to $\mathcal{M}$, and it is pure.

Proof of $\Leftarrow$ in KS theorem

If $\psi : A \to \mathbb{C}$ is a pure state, then it is a (unital) algebra $*$-morphism, so $\psi(\mathcal{P}(A)) = \{0, 1\}$.

Claim: If φ is a state on $\mathcal{M}$ extending ψ then A is in the centralizer of φ, i.e. $\varphi(yx) = \varphi(xy)$, $\forall x \in A$, $y \in \mathcal{M}$. Sufficient to prove for $y = p \in \mathcal{P}(A)$ with $\psi(p) = 0$ (because it holds for $y = 1$). But by C-S inequality $|\varphi(xp)| \leq \varphi(xx^*)^{1/2}\psi(p)^{1/2} = 0$ and similarly $\varphi(px) = 0$.

Thus, $\varphi(x) = \varphi(\Sigma_k p_k x p_k)$, $\forall x \in \mathcal{M}$, $\forall \{p_k\}_k \subset \mathcal{P}(A)$ finite partition of 1. Taking limits, we get $\varphi(x) = \varphi(E(x)) = \psi(E(x))$.

This shows that: $\psi \circ E$ is the unique state extension of ψ to $\mathcal{M}$ (and therefore $\psi \circ E$ pure); E is a conditional expectation, and it is unique.

Proof of $\Rightarrow$ in KS theorem

Let $b = b^* \in \mathcal{M}$ and fix $t \in \Omega$ where $A = C(\Omega)$. Denote $\gamma_0 = \inf\{a(t) \mid a = a^* \in A, a \geq b\}$, $\gamma_1 = \sup\{a(t) \mid a = a^* \in A, a \leq b\}$. We first show that the hypothesis implies $\gamma_0 = \gamma_1$.

For if not, then the maps $\psi_i : A + \mathbb{C}b \to \mathbb{C}$ defined by $\psi_i(y + \alpha b) = y(t) + \alpha\gamma_i$, $i = 0, 1$, $y \in A$, $\alpha \in \mathbb{C}$, are well defined, linear and positive; thus $\|\psi_i\| = 1$ and by Hahn-Banach each ψ_i can be extended to a norm-1 linear functional $\varphi_i : \mathcal{M} \to \mathbb{C}$; we have thus obtained two states φ_0, φ_1 on $\mathcal{M}$, which extend the pure state t and are distinct (because $\varphi_0(b) \neq \varphi_1(b)$), contradicting the assumption. Thus, $\gamma_0 = \gamma_1$.

Let now $\varepsilon > 0$ and for each $t \in \Omega$ denote

$$c_t = \inf\{a(t) \mid a = a^* \in A, a \geq b\} = \sup\{a(t) \mid a = a^* \in A, a \leq b\}.$$

Let $a_t^{\pm} \in A$ be selfadjoint elements such that $a_t^+ \geq b \geq a_t^-$ and $c_t + \varepsilon/2 > a_t^+(t)$, $a_t^-(t) > c_t - \varepsilon/2$.

Proof of $\Rightarrow$ in KS theorem (continuation)

By the continuity of $a_t^{\pm} \in A = C(\Omega)$ as a function on Ω, there exists an open-closed neighborhood Ω_t of t in Ω such that

$$c_t + \varepsilon > a_t^+(t'), a_t^-(t') > c_t - \varepsilon, \forall t' \in \Omega_t.$$

Thus, if $p_t \in C(\Omega)$ is the characteristic function of Ω_t, then $p_t \in \mathcal{P}(A)$ satisfies

$$(c_t + \varepsilon)p_t \geq a_t^+ p_t \geq p_t b p_t \geq a_t^- p_t \geq (c_t - \varepsilon)p_2.$$

In particular, $\|p_t b p_t - c_t p_t\| \leq \varepsilon$. Since Ω is compact, there exist $t_1, ..., t_n \in \Omega$ such that $\cup_i \Omega_{t_i} = \Omega$. If we now take q_1 to be the characteristic function of Ω_{t_1} and for each $j \geq 2$, p_j to be the characteristic function of $\Omega_j \setminus \cup_{i=1}^{j-1}\Omega_i$, viewed as a projection in A, it follows that $\|\Sigma_j q_j b q_j - \Sigma_j c_{t_j} q_j\| \leq \varepsilon$.

$\square$

Conclusions about MASAs in $\mathcal{B}(\ell^2\mathbb{Z})$ (K-S 1959)

• K-S went on and proved that $L(\mathbb{Z}) \subset \mathcal{B}(\ell^2\mathbb{Z})$ doesn't satisfy the KS property (equivalently, the paving property), by showing that there exist two distinct conditional expectations from $\mathcal{B}(\ell^2\mathbb{Z})$ onto $L(\mathbb{Z})$ (we saw that uniqueness of c.e. is a prerequisite for KS property to hold).

• K-S have noticed that the map E that assigns to $[x_{ij}] \in \mathcal{B}(\ell^2\mathbb{Z})$ its diagonal $[x_{ii}\delta_{ij}] \in \ell^\infty\mathbb{Z}$, is the unique conditional expectation of $\mathcal{B}(\ell^2\mathbb{Z})$ onto $\ell^\infty\mathbb{Z}$ and that each "vector pure state" on $\ell^\infty\mathbb{Z}$ has unique state extension. But they were not able to settle the case of arbitrary (singular) pure states, thus leaving the KS property for the diagonal, atomic MASA $\ell^\infty\mathbb{Z}$ as an open problem. Yet they expressed the belief that the problem has a negative answer !

The Classic Kadison-Singer Problem:

Is it true that any pure state on $\ell^\infty\mathbb{Z}$ extends to a unique (pure) state on $\mathcal{B}(\ell^2\mathbb{Z})$ (i.e. $\ell^\infty\mathbb{Z} \subset \mathcal{B}(\ell^2\mathbb{Z})$ has the KS property)? Equivalently, does $\ell^\infty\mathbb{Z} \subset \mathcal{B}(\ell^2\mathbb{Z})$ have the paving property?

Finite dimensional reformulations

• Anderson 1978: Paving holds iff *uniform paving* holds: $\forall \varepsilon > 0$, $\exists n = n(\varepsilon)$ such that $\forall x \in \mathcal{B}(\ell^2\mathbb{Z})$ with 0 on the diagonal, $\exists p_1, ..., p_n \in \ell^\infty\mathbb{Z}$ with $\Sigma_k p_k = 1$ and $\|\Sigma_k p_k x p_k\| \leq \varepsilon \|x\|$.
Proof: by taking direct sum of operators...

• Anderson 1979: Uniform paving holds iff *uniform finite dimensional paving* holds: $\forall \varepsilon > 0$, $\exists n = n(\varepsilon)$ such that $\forall m$ and $\forall x \in M_{m\times m}(\mathbb{C})$ with 0 on the diagonal D_m, $\exists p_1, ..., p_n \in \mathcal{P}(D_m)$ partition of 1 satisfying $\|\Sigma_k p_k x p_k\| \leq \varepsilon \|x\|$. In fact, it is sufficient to prove this for some $\varepsilon_0 < 1$.

Notation Given a MASA in a vN algebra $A \subset \mathcal{M}$, we denote by $n(A \subset \mathcal{M}; \varepsilon)$ the minimum over all n with the property that $\forall x \in \mathcal{M}$, $\exists p_1, ..., p_n \in \mathcal{P}(A)$ with $\Sigma p_k = 1$ and $d(\Sigma p_k x p_k, A) \leq \varepsilon d(x, A)$ and call it the ε-*paving size of* $A \subset \mathcal{M}$.

• With this notation, Anderson's 1979 result actually reads: $n(\ell^\infty\mathbb{Z} \subset \mathcal{B}(\ell^2\mathbb{Z}); \varepsilon) = \sup_m n(D_m \subset M_{m\times m}(\mathbb{C}); \varepsilon)$, $\forall \varepsilon > 0$.
Proof: $\geq$ clear and $\leq$ by taking weak limits of partitions in D_m

Further developments 1980-2012

• Berman-Halpern-Kaftal-Weiss 1987: any matrix with non-negative entries can be paved; also any Riemann-integrable function on $\mathbb{T}$, viewed (via Fourier expansion) as an element in $L(\mathbb{Z}) \simeq L^\infty(\mathbb{T})$ acting on $\ell^2\mathbb{Z} \simeq L^2(\mathbb{T})$ as left convolution/multiplication operator, can be paved.

• Bourgain-Tzafriri 1991: Paving of elements in $L(\mathbb{Z}) \simeq L^\infty(\mathbb{T})$ with Fourier expansion satisfying certain growth properties. Paving of multiplication operators in $L^\infty(\mathbb{T})$ became a famous conjecture in harmonic analysis.

• Akemann-Anderson 1994: To prove the unif. fin. dim. paving conjecture it is sufficient to prove: $\exists \delta > 0$ and $c < 1$ such that for all m and all $q \in \mathcal{P}(M_{m\times m}(\mathbb{C}))$ with diagonal entries $\leq \delta$, $\exists p \in \mathcal{P}(D_m)$, satisfying $\|pqp + (1-p)q(1-p)\| \leq c$. (Proof: dilation trick)

• An equivalent reformulation in frame theory (the Feichtinger conjecture); much interest in applied math and engineering.

• During 2000-2012 much work by Weaver, Paulsen&collaborators, Casazza&collaborators, etc

The Marcus-Spielman-Srivastava (MSS) solution to the classic KS problem, June 2013

Theorem (MSS: math.OA/1306.3969)

If $\delta > 0$ is given, then for any m and any projection $q \in M_{m\times m}(\mathbb{C})$ with all entries on the diagonal $\leq \delta$, there exists a projection $p \in D_m$ such that

$$\|pqp + (1-p)q(1-p)\| \leq (1+\sqrt{2\delta})^2/2.$$

Proof Very ingenious estimations of norms of matrices, by estimating the largest roots of the corresponding characteristic polynomials (method of *interlacing polynomials*).

- Entails $n(\ell^\infty\mathbb{Z} \subset \mathcal{B}(\ell^2\mathbb{Z}); \varepsilon) = \sup_m n(D_m \subset M_{m\times m}(\mathbb{C}); \varepsilon) \leq C\varepsilon^{-6}$.

II_1 factor formulations of the KS problem

Two simple facts (P: 1997, resp. 2009/march 2013)

1. If A is a separable MASA in a II_1 factor M, then there are non-normal conditional expectations of M onto A, thus $A \subset M$ cannot satisfy the KS property nor the paving property.
2. Let ω be a free ultrafilter. Denote D the Cartan "diagonal" subalg. of the hyperfinite II_1 factor R. The classic KS is equivalent to the KS for the (non-separable) MASAs $D^\omega \subset R^\omega$, resp. $\Pi_\omega D_m \subset \Pi_\omega M_{m\times m}(\mathbb{C})$. More precisely, both these inclusions have the same paving size as $\ell^\infty\mathbb{Z} \subset \mathcal{B}(\ell^2\mathbb{Z})$, and thus equal to $\sup_m n(D_m \subset M_{m\times m}(\mathbb{C}))$.

Corollary to MSS result

$D^\omega \subset R^\omega$, $\Pi_\omega D_m \subset \Pi_\omega M_{m\times m}(\mathbb{C})$ satisfy the KS property, i.e. any pure state on D^ω (resp. $\Pi_\omega D_m$) extends to a unique (pure) state on R^ω (resp. $\Pi_\omega M_{m\times m}(\mathbb{C})$). Moreover, the order of magnitude of the paving size of both these inclusions is $\leq \varepsilon^{-6}$. Also, the trace preserving expectation is the unique expectation.

KS property for ultraproducts of singular MASAs

- Recall that if A is a MASA in a II_1 factor M, then $\mathcal{N}_M(A)$ denotes the *normalizer* of A in M, $\{u \in \mathcal{U}(M) \mid uAu^* = A\}$. The MASA $A \subset M$ is *singular* if $\mathcal{N}_M(A) = \mathcal{U}(A)$ (Dixmier 1954). A typical example is $L(\mathbb{Z}) \subset L^\infty([0,1]^{\mathbb{Z}}) \rtimes \mathbb{Z} = R$ (P 1981). But in fact any separable II_1 factor M has singular MASAs (P 1981).

Thm (P: math.OA/1303.1424)

Let $A_m \subset M_m$, $m \geq 1$, be singular MASAs in II_1 factors and denote $\mathbf{A} = \Pi_\omega A_m \subset \Pi_\omega M_m = \mathbf{M}$, their ultraproduct, over a free ultrafilter ω. Then $\mathbf{A} \subset \mathbf{M}$ satisfies the KS property, i.e. any pure state on $\mathbf{A}$ has a unique state extension to $\mathbf{M}$. Moreover, the order of magnitude of the paving size of $\mathbf{A} \subset \mathbf{M}$ is $\leq \varepsilon^{-6}$. Also, the trace preserving expectation is the unique expectation of $\mathbf{M}$ onto $\mathbf{A}$.

About the proof

We first show that it is sufficient to pave $e \in \mathcal{P}(\mathbf{M})$ with $E_{\mathbf{A}}(e) = \tau(e)1$.

Key part is to show that $\forall B \subset \mathbf{M}$ separable with $B \perp \mathbf{A}$, $\exists u \in \mathcal{U}(\mathbf{A})$ Haar s.t. $\tau(\Pi_{i=1}^k x_i u^{n_i}) = 0$, $\forall k$, $\forall x_i \in B \ominus \mathbb{C}1$, $n_i \neq 0$ for all i but possibly one.

Thus, B and $B_0 = \{u\}''$ generate $B * B_0$. (Obs: By Kesten 1959, this already implies $\forall v \in \mathcal{U}(B)$ Haar has *free pavings*: if $p_1, ..., p_n \in \mathcal{P}(B_0)$ partition with $\tau(p_i) = 1/n$, then $\|\Sigma_{i=1}^n p_i v p_i\| = 1/\sqrt{n}$.)

More importantly, by Voiculescu's 1985 norm calculation for products of free independent projections q, f with $\tau(q) \leq \tau(f) \leq 1/2$ one has:

$$\|q(f - \tau(f)1)q\| = \tau(q) - 2\tau(f)\tau(q) + \sqrt{4\tau(f)(1-\tau(f))\tau(q)(1-\tau(q))}$$

By applying this to $B = \mathbb{C}e + \mathbb{C}(1-e)$ and to partitions $p_1, ..., p_n \in B_0$ with $n \geq \tau(e)^{-1}$, one obtains the desired pavings of e:

$$\|\Sigma_i p_i e p_i - \tau(e)1\| \leq 2/\sqrt{n}.$$

Note: if $p \in \mathcal{P}(B_0)$ with $\tau(p) = 1/2$ and $e \in \mathcal{P}(B)$, $\tau(e) = \delta \leq 1/2$, then

$$\|pep + (1-p)e(1-p)\| = 1/2 + \sqrt{\delta(1-\delta)}.$$

Characterizations of singularity for MASAs

Question: Can one find such (asymptotic) free independence in arbitrary MASAs $A \subset M$? No, because if $v \in \mathcal{N}_M(A)$ then $vuv^*u^* = u^*vuv^*$, $\forall u \in \mathcal{U}(A^\omega)$, so $\tau(vuv^*u^*vu^*v^*u) = 1$. Thus, 4-independence fails for non-singular MASAs (where for $n \geq 1$, $B_0 \subset A^\omega$ a subalgebra and $X \subset M^\omega \ominus A^\omega$ a subset, we say that B_0 is n-*independent to* X if $\tau(\Pi_{i=1}^k u_i x_i) = 0$, $\forall k \leq n$, $u_i \in B_0 \ominus \mathbb{C}$, $x_i \in X$). We actually have:

Thm (P: math.OA/1303.1424)

Let A be a MASA in a II_1 factor M. Given any separable subset $X \subset M^\omega \ominus A^\omega$, there exists a diffuse subalgebra $B_0 \subset A^\omega$ such that B_0 is 3-independent to X. Moreover, the following are equivalent for A:

1° $\forall x \in M \ominus A$, $\exists B_0 \subset A^\omega$ diffuse such that B_0 is 4-independent to $\{x\}$.

2° A is singular in M.

3° Given any $X \subset M^\omega \ominus A^\omega$ separable, there exists $B_0 \subset A^\omega$ diffuse such that B_0 is free independent to X.

4° A^ω is maximal amenable in M^ω.