“Chiral Quantum Cloning”

– Representation theory, spectral invariants and symmetries in algebraic conformal quantum field theory –

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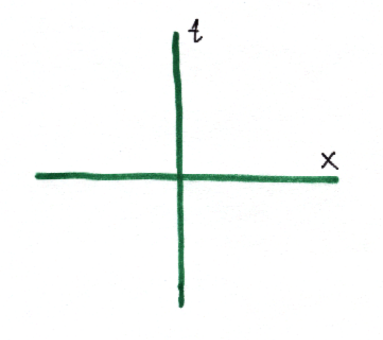
What this talk is about

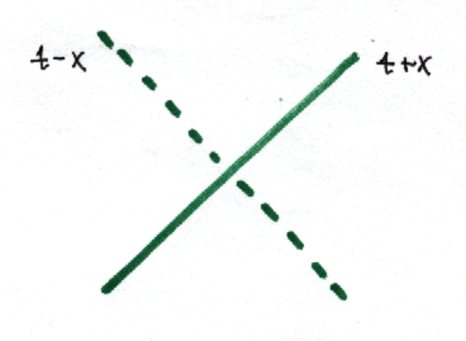
PRELUDIO

The world ...



... is two-dimensional.

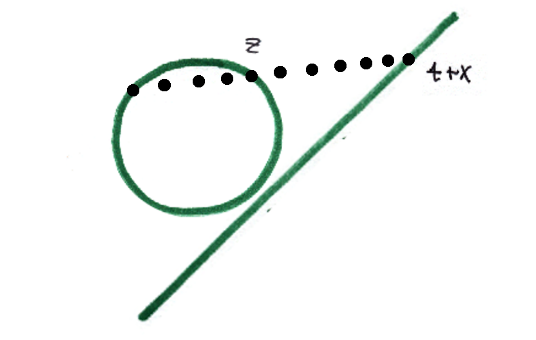
The chiral world . . .



. . . is one-dimensional.

"Time is Space", 1 lightyear = 9.450.000.000.000.000 m

The projective chiral world ...



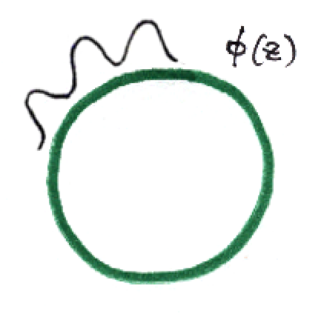
... is a circle.

The most important physical observables in Conformal QFT are

CHIRAL QUANTUM FIELDS.

Energy and momentum densities, charge densities, . . .

Chiral quantum fields . . .

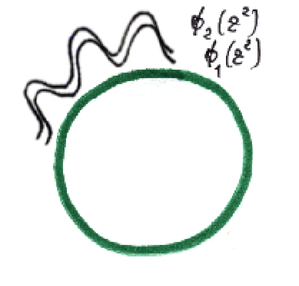


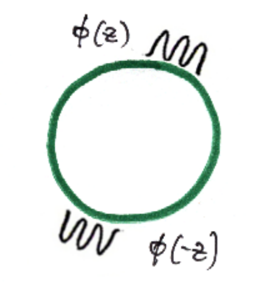
. . . are assigned to points or intervals on the circle.

“Chiral quantum cloning”

Roberto’s canonical isomorphism

Two copies of a quantum field in one interval = . . .





. . . = one quantum field in two disjoint intervals.

[Longo-Xu ’04, Kawahigashi-Longo ’05]

What is such an isomorphism possibly good for?

Applications:

- Study of representation theory
- NCG spectral invariants
- Multilocal symmetries
- Modular theory of two-intervals
- C_2 cofiniteness (?)

TEMA CON VARIAZIONI

The general setting

AQFT on the circle

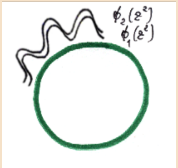
- Localized von Neumann algebras $A(I)$
- Local commutativity
- Diffeomorphism ($=$ conformal) covariance
- Vacuum and positive-energy representations

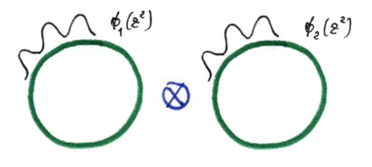
Recall: The split property

- ... states that the map $a \otimes b \mapsto ab$ for $a \in A(I_1)$ and $b \in A(I_2)$ is an isomorphism of von Neumann algebras $A(I_1) \otimes A(I_2)$ and $A(I_1) \vee A(I_2) \equiv A(I_1 \cup I_2)$ (whenever I_1 and I_2 do not touch).
- ... expresses the possibility of independent preparations of partial states on the subalgebras $A(I_1)$ and $A(I_2)$.
- ... follows from a decent growth of the dimensions of the eigenspaces of L_0 (namely, $e^{-\beta L_0}$ should be trace-class in the vacuum representation, Buchholz-Wichmann 1986).
- ... is assumed throughout.

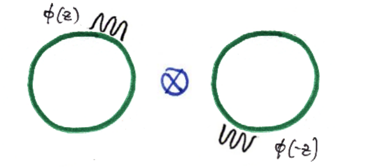
Roberto's canonical isomorphism

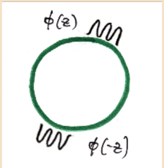
. . . composes diffeomorphisms $z \to \pm\sqrt{z}$ with the split isomorphism:





$$A^{\otimes 2}(I) \equiv A(I) \otimes A(I) \xrightarrow{\mathrm{Diff}\otimes\mathrm{Diff}} A(I_1) \otimes A(I_2) \xrightarrow{\text{Split iso}} A(I_1) \vee A(I_2)$$





[Longo-Xu '04, Kawahigashi-Longo '05]

Chiral quantum cloning

$$\iota_I = \chi_{I_1 \cup I_2} \circ (\delta_{\sqrt{\cdot}} \otimes \delta_{-\sqrt{\cdot}}) : A^{\otimes 2}(I) \longrightarrow A(I_1 \cup I_2),$$

where $I_1 \cup I_2 \equiv \sqrt{I}$ and $A^{\otimes 2}(I) \equiv A(I) \otimes A(I)$.

Some facts to memorize:

- These isomorphisms do not preserve the ground state (vacuum) because the tensor product suppresses all correlations, and because diffeomorphisms do not preserve the vacuum.
- They are defined for each interval I, but they are compatible as I increases.
- They extend to the entire circle minus a point.
- Everything generalizes to n copies, n intervals, using $\sqrt[n]{\cdot}$.

Mode-doubling

- For Virasoro and affine Kac-Moody algebras it is well-known that
$$\widehat{L}_n = \frac{1}{2} L_{2n} + \frac{c}{16}\delta_{n0} \qquad \text{and} \qquad \widehat{J}^a_n = \frac{1}{2} J^a_{2n}$$
satisfy the same commutation relations as L_n and J^a_n (with central charge $2c$ resp. level $2k$).
- This is “one half” of the canonical isomorphism, applying ι to $T(z^2) \otimes \mathbf{1} + \mathbf{1} \otimes T(z^2)$ resp. $J^a(z^2) \otimes \mathbf{1} + \mathbf{1} \otimes J^a(z^2)$.
- The “other half” requires half-integer modes, ie, is twisted.

VAR 1: Representation theory

Longo-Xu (2004)

The canonical representation

- The canonical isomorphisms

$$\iota_I = \chi_{I_1 \cup I_2} \circ (\delta_{\sqrt{\cdot}} \otimes \delta_{-\sqrt{\cdot}}) : A^{\otimes 2}(I) \to A(I_1 \cup I_2)$$

  extend to a representation π of $A^{\otimes 2}$ on $S^1 \setminus \{-1\} = \mathbb{R}$.
- The extension to the entire circle S^1 is not possible. Instead π differs on both sides of $z = -1$ by the flip $a \otimes b \mapsto b \otimes a$.
- $\Rightarrow \pi$ is a soliton (or twisted) representation of $A^{\otimes 2}(\mathbb{R})$.
- π restricts to a true (DHR) representation π_B of the flip-invariant subnet B on $\mathbb{R}$.
- π_B is reducible with exactly two irreducible components.

Recall: Dimension and μ-index

- The dimension d_π (possibly ∞) of a representation is the square root of the index of the subfactor $\pi(A(I)) \subset \pi(A(I'))'$. [Longo 1989]
- The μ-index (possibly ∞) of a chiral CFT on S^1 is the index of the two-interval subfactor $A(I_1 \cup I_2) \subset A((I_1 \cup I_2)')'$ in the vacuum representation.
- The μ-index equals the "global index"

$$\mu_A = \sum_\pi d_\pi^2.$$

[Kawahigashi-Longo-Müger 2001]

The Longo-Xu dichotomy

The CMS property:
"A has at most countably many sectors, all with finite dimension."

- If A has the CMS property, then $A^{\otimes 2}$ and $B \subset A^{\otimes 2}$ have the CMS property.
- In this case, $\pi_B = \pi_1 \oplus \pi_2$ has finite dimension.
- Because the index of $B(I) \subset A^{\otimes 2}(I)$ is two, it follows that π has finite dimension.
- But $d_\pi^2 =$ the index of $\pi_I(A^{\otimes 2}(I)) \subset \pi_{I'}(A^{\otimes 2}(I'))'$ equals the two-interval μ-index of the net A. Thus, μ_A is finite.
- $\Rightarrow$ **If A has at most countably many sectors, then either the number of sectors is actually finite, or some sector has infinite dimension.**
- The first possibility also implies strong additivity, hence complete rationality.

VAR 2: NCG spectral invariants

Kawahigashi-Longo (2005)

Recall: Modular theory

Given a von Neumann algebra M and a faithful normal state $\varphi = (\Phi, \cdot\Phi)$, the polar decomposition of the antilinear operator $m\Phi \mapsto m^*\Phi$ gives rise to an anti-involution J and a unitary 1-parameter group Δ^{it} such that

$$JMJ = M', \qquad \sigma_t := Ad_{\Delta^{it}}|_M \subset Aut(M).$$

- The modular group σ_t is an "intrinsic dynamics" of M, depending only on the state.
- For $M = A(S^1_+)$ and $\Phi = \Omega$, one has $\Delta^{it} = e^{-2\pi(L_1 - L_{-1})t} =$ scale transformation of $\mathbb{R}_+$ (Bisognano-Wichmann property, geometric modular action), and $J =$CPT (Brunetti-Guido-Longo 1993).
- In particular, the Möbius group (in 4D: the Poincaré group) is "of modular origin".

The Kawahigashi-Longo state

- Non-commutative elliptic geometry interpretation of "modular" free energy and entropy.
- n intervals as $n \to \infty$ unite discrete and conformal features of NCG.
- Consider the state
$$\varphi_n = \omega^{\otimes n} \circ \iota_n^{-1}$$
on $A(\sqrt[n]{I})$, where ι_n is the (n-interval) canonical isomorphism, and $\omega^{\otimes n} = \omega \otimes \cdots \otimes \omega$ the vacuum state of $A^{\otimes n}(I)$.
- Its modular group is generated by $\frac{-2\pi}{n}(L_n - L_{-n})$. Hence the entire diffeomorphism symmetry is of modular origin.
- One may extract spectral invariants ("entropies"). In particular, the μ-index arises as a spectral invariant.
- A chiral CFT is expected to live on the horizon of a Black Hole: Relation to Bekenstein entropy.

VAR 3: Multilocal symmetries

KHR-Tedesco (2013)

Multilocal currents

- If A is the real free Fermi CFT, then $A^{\otimes 2}$ is the complex free Fermi CFT which has an $SO(2) = U(1)$ local gauge symmetry
$$\gamma : (\psi_1 + i\psi_2)(z) \mapsto e^{i\alpha(z)}(\psi_1 + i\psi_2)(z).$$
- The generator of the gauge symmetry is a free Bose current affiliated with $A^{\otimes 2}(I)$.
- The canonical isomorphism maps the current J into the two-interval real Fermi algebra $A(I_1 \cup I_2)$. The embedded current is **"multi-local"**:
$$\iota(J(z^2)) = \frac{1}{2z} :\psi(z)\psi(-z): .$$

Multilocal symmetries

- The multilocal current $\frac{1}{2z} :\psi(z)\psi(-z):$ generates transformations of the real Fermi field

$$\gamma_m = \iota \circ \gamma \circ \iota^{-1}.$$

- γ_m are **multilocal symmetries** $=$ z-dependent $SO(2)$ "mixings" between $\psi(z)$ and $\psi(-z)$.
- Similar with the stress-energy tensor of $A^{\otimes 2}$ ($=$ generator of diffeomorphisms of $A^{\otimes 2}$). Embedded into $A(I_1 \cup I_2)$, it generates **diffeomorphisms plus mixing**.

VAR 4: Modular theory for 2-intervals

Longo-Martinetti-KHR (2011),
in the light of “multilocal symmetries”.

Vacuum-preserving isomorphism

- Recall: The canonical isomorphism ι does NOT preserve the vacuum.
- For the free Fermi theory (CAR), we found another global and **vacuum preserving isomorphism** $\beta : A^{\otimes 2}(I) \to A(I_1 \cup I_2)$.
- Remarkably, β and ι just differ by a gauge transformation of $A^{\otimes 2}$:

$$\beta = \iota \circ \gamma.$$

- Hence, the Kawahigashi-Longo state $\varphi_2 := \omega^{\otimes 2} \circ \iota^{-1}$ equals $\omega \circ \beta \circ \iota^{-1} = \omega \circ \gamma_m$, where $\gamma_m = \iota \circ \gamma \circ \iota^{-1}$ is a multilocal gauge transformation (=mixing) of the two-interval algebra.
- The gauge transformation γ absorbs the twist of ι, so that β extends to the entire S^1.

Two-interval modular group

- Because $\beta = \iota \circ \gamma = \gamma_m \circ \iota$ preserves the vacuum state, it intertwines the vacuum modular groups of $A^{\otimes 2}(I)$ and $A(I_1 \cup I_2)$.
- This allows to compute the **modular group for the two-interval Fermi algebra in the vacuum state**

$$\sigma_t^{(2)} = \gamma_m \circ \iota \circ \sigma_t \circ \iota^{-1} \circ \gamma_m^{-1}.$$

- The canonical isomorphism ι maps $\sigma_t \in$ Möbius group to the 2-Möbius group, and the gauge transformation γ contributes a mixing γ_m of the intervals.
- The modular group "acts geometrically plus mixing":

The Connes cocycle

- Reproduces "in one line" the intricate explicit computation of the two-interval modular group by Casini and Huerta (2011).
- Also allows to compute the **Connes cocycle** between the vacuum state and the Kawahigashi-Longo state φ_2 on the 2-interval algebra:
- Let $W(f) = e^{iJ(f)}$ be a Weyl operator $\in A^{\otimes 2}(\tilde{I})$ ($\tilde{I} \supset I$) that implements the gauge transformation γ inside I. Let δ_t the geometric one-interval modular action. Then $f - f \circ \delta_t$ is zero at the endpoints of the interval, and $g_t := 1_I \cdot (f - f \circ \delta_t)$ is continuous with support in I. Hence $W(g_t) \in A^{\otimes 2}(I)$. The Connes cocycle turns out to be

$$(D\omega : D\varphi_2)_t = \iota_I(W(g_t)) \in A(I_1 \cup I_2).$$

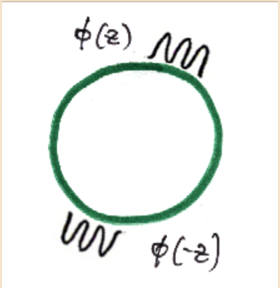
CODA

Outlook

- Unlike the canonical isomorphism ι, the vacuum-preserving isomorphism β does NOT descend to subalgebras, such as Virasoro or current, of the free Fermi theory.
- $\Rightarrow$ the two-interval modular groups for these theories remain unknown.
- Expectation: β should exist for all models without sectors (global index $\mu_A = 1$)
- (…many open questions …)
- Formulation in terms of Vertex Operator Algebras ("mode-doubling")?
- Relation to C_2-cofiniteness?

IL FINE, and ...

(another iso)



HAPPY YEARS TO COME, ROBERTO!