Operator-algebraic construction of two-dimensional quantum field models

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Introduction

Formulation of quantum field theory

- Wightman fields
- Osterwalder-Schrader axioms
- **operator-algebraic approach** (Haag-Kastler nets)

Main open problem of this field

No nontrivial example in 4 spacetime dimensions.

Recent progress in operator-algebraic approach:

- reconstruction of net from a single von Neumann algebra and the Tomita-Takesaki theory (Borchers '92)
- factorizing scalar S-matrix models (Lechner '08)

Present approach:

- purely **operator algebraic** construction of nets (cf. Longo-Witten '11)

Main result

Interacting Haag-Kastler nets in 2 dim, and more partial constructions.

Quantum fields

Wightman axioms

- ϕ: operator-valued distribution on $\mathbb{R}^d$, $[\phi(x), \phi(y)] = 0$ if $x \perp y$ (**observable at** x)
- U: the spacetime symmetry, $U(g)\phi(x)U(g)^* = \phi(gx)$
- Ω the vacuum vector

Equivalently, one considers n-point functions (Wightman functions)

$$W(x_1, t_1, x_2, t_2, \cdots, x_n, t_n) = \langle \Omega, \phi(x_1, t_1)\phi(x_2, t_2) \cdots \phi(x_n, t_n)\Omega \rangle.$$

or their Wick-rotations $S(\cdots x_k, t_k \cdots) := W(\cdots x_k, it_k \cdots)$ (Schwinger functions).

- examples in 2 and 3 dimensions (Glimm, Jaffe, ...)
- $\phi(f)$ is an unbounded operator

Net of von Neumann algebras

Haag-Kastler net

$\mathcal{A}(O)$: **von Neumann algebras** (weakly closed algebras of bounded operators on a Hilbert space $\mathcal{H}$) parametrized by open regions $O \Subset \mathbb{R}^d$

- isotony: $O_1 \subset O_2 \Rightarrow \mathcal{A}(O_1) \subset \mathcal{A}(O_2)$
- locality: $O_1 \perp O_2 \Rightarrow [\mathcal{A}(O_1), \mathcal{A}(O_2)] = 0$
- Poincaré covariance: $\exists U$: positive energy rep of $\mathcal{P}_+^\uparrow$ such that $U(g)\mathcal{A}(O)U(g)^* = \mathcal{A}(gO)$
- vacuum: $\exists \Omega$ such that $U(g)\Omega = \Omega$ and cyclic for $\mathcal{A}(O)$

Correspondence: $\mathcal{A}(O) = \{e^{i\phi(f)} : \mathrm{supp} f \subset O\}''$
(**observables measured in** O)

Borchers triples

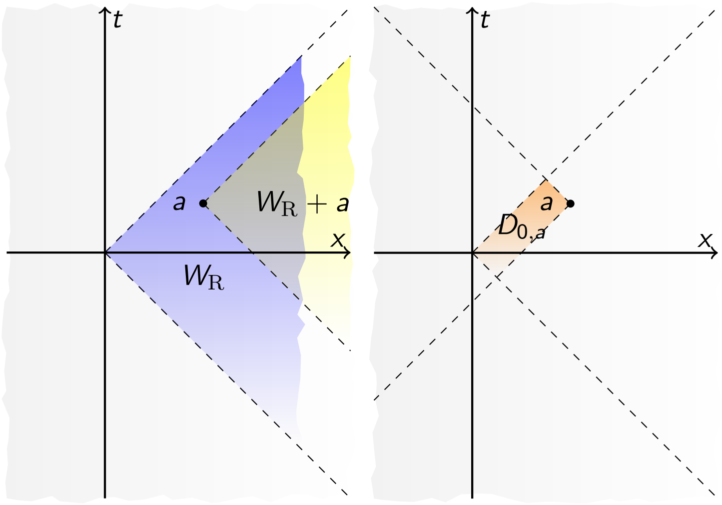
Infinitely many von Neumann algebras $\Rightarrow$ difficult to construct nets. **Borchers triple** reduces the question to a **single** von Neumann algebra if the spacetime has **dimension 2**.

- Haag-Kastler net: von Neumann algebras $\mathcal{A}(O)$ parametrized by open regions O acted on by the Poincaré group
- Borchers triple: a **single** von Neumann algebra $\mathcal{M}$ acted on by spacetime **translations**

Idea (net $\Longrightarrow$ Borchers triple): consider only **wedges** $\mathcal{A}(W_{\mathrm{R}})$, $W_{\mathrm{R}} := \{a = (a_0, a_1) : |a_0| < a_1\}$.
The net $\mathcal{A}$ can be recovered from wedges (Borchers '92):

$$\mathcal{A}(D_{a,b}) = (U(a)\mathcal{A}(W_{\mathrm{R}})U(a)^*) \cap (U(b)\mathcal{A}(W_{\mathrm{R}})'U(b)^*).$$

Standard wedge and double cone



Borchers triples

Definition

$(\mathcal{M}, T, \Omega)$, where $\mathcal{M}$: vN algebra, T: positive-energy rep of $\mathbb{R}^2$, Ω: vector, is a Borchers triple if Ω is cyclic and separating for $\mathcal{M}$ and

- $\mathrm{Ad}\, T(a)(\mathcal{M}) \subset \mathcal{M}$ for $a \in W_{\mathrm{R}}$, $T(a)\Omega = \Omega$

Borchers triple $\Longrightarrow$ net

If one defines a "net" by $\mathcal{A}(D_{a,b}) := (U(a)\mathcal{M}U(a)^*) \cap (U(b)\mathcal{M}'U(b)^*)$, then T can be extended to a rep U of Poincaré group and satisfies all the axioms of local net **except the cyclicity of vacuum**.

Problems

- to construct new Borchers triples (wedge-local QFT)
- to show the cyclicity of vacuum (strict locality)

Internal symmetry

An **internal symmetry** on a Borchers triple $(\mathcal{M}, T, \Omega)$ is a unitary representation W of a group G such that $\mathrm{Ad}\,V(g)\mathcal{M} = \mathcal{M}$, $[V(g), T(a)] = 0$ and $V(g)\Omega = \Omega$.
We take an action V of S^1. $V(t) = e^{itQ}$. $\widetilde{V}(t) = e^{itQ\otimes Q}$.

Theorem (T. arXiv:1301.6090)

Let $\widetilde{\mathcal{M}}_t = (\mathcal{M}\otimes \mathbb{1}) \vee \mathrm{Ad}\,\widetilde{V}(t)\,(\mathbb{1}\otimes\mathcal{M})$, $\widetilde{T}(a) = T(a)\otimes T(a)$, $\widetilde{\Omega} = \Omega\otimes\Omega$. Then $(\widetilde{\mathcal{M}}_t, \widetilde{T}, \widetilde{\Omega})$ is a Borchers triple.

Proof: $\widetilde{V}(t) = \sum_k V(kt)\otimes dE(k)$.
$\widetilde{\mathcal{M}}'_t = \mathrm{Ad}\,\widetilde{V}(t)\,(\mathcal{M}'\otimes\mathbb{1}) \vee (\mathbb{1}\otimes\mathcal{M}')$.

Question: is $(\widetilde{\mathcal{M}}_t, \widetilde{T}, \widetilde{\Omega})$ **strictly local**?
Lemma (Lechner '08): If there is a type I factor ($\cong B(\mathcal{H})$) between $\mathrm{Ad}\,T(a)(\widetilde{\mathcal{M}}_t) \subset \widetilde{\mathcal{M}}_t$, $a \in W_{\mathrm{R}}$ (**wedge-split inclusion**), then strict locality follows.

Strictly local nets

Let $(\mathcal{M}, T, \Omega)$ be a wedge-split Borchers triple with S^1 action V.

Theorem (T. arXiv:1301.6090)

The triple $(\widetilde{\mathcal{M}}_t, \widetilde{T}, \widetilde{\Omega})$ is wedge-split, hence strictly local.

Proof: an intermediate type I factor is given by
$\widetilde{\mathcal{R}}(a) = (\mathcal{R}(a) \otimes \mathbb{1}) \vee \mathrm{Ad}\, V(t)(\mathbb{1} \otimes \mathcal{R}(a))$, where $\mathrm{Ad}\, T(a)(\mathcal{M}) \subset \mathcal{R}(a) \subset \mathcal{M}$ and $\mathcal{R}(a)$ is the canonical intermediate type I factor (Doplicher-Longo '84).

Example: the complex massive free net $(\mathcal{M}, T, \Omega) \Rightarrow (\widetilde{\mathcal{M}}_t, \widetilde{T}, \widetilde{\Omega})$ has nontrivial S-matrix (**interacting**).

More examples:

- $(\widetilde{\mathcal{M}}_t, \widetilde{T}, \widetilde{\Omega})$ is again wedge-split with internal symmetry.
- $(\mathcal{N} \otimes \mathcal{N}, T \otimes T, \Omega \otimes \Omega)$ where $(\mathcal{N}, T, \Omega)$ is one of nets with factorizing S-matrix (Lechner '08).

Conjecture: $P(\phi)_2$ models have wedge-split property?

More Borchers triples

Let $(\mathcal{M}, T, \Omega)$ be from the real **massive free field**.

Take generators $T(a_+, 0) = e^{ia_+P_+}$ and set $\widetilde{R}_\varphi = e^{it\frac{1}{P_+}\otimes P_+}$, $t > 0$.

Theorem (T. arXiv:1301.6090)

Let $\widetilde{\mathcal{M}}_t = (\mathcal{M}\otimes \mathbb{1}) \vee \mathrm{Ad}\widetilde{R}_\varphi(\mathbb{1}\otimes\mathcal{M})$, $\widetilde{T}(a) = T(a)\otimes T(a)$, $\widetilde{\Omega} = \Omega\otimes\Omega$.
Then $(\widetilde{\mathcal{M}}_t, \widetilde{T}, \widetilde{\Omega})$ *is a Borchers triple.*

Many more examples with inner symmetric function φ (cf. Longo-Witten '11). This is the simplest case $\varphi(p) = e^{itp}$.

Strict locality: by showing modular nuclearity? (cf. Lechner '08).

How a general R_φ looks like...

For an inner symmetric function φ, set

- $\mathcal{H}^n := \mathcal{H}_1^{\otimes n}$
- $P_{i,j}^{m,n} := (\mathbb{1} \otimes \cdots \otimes \underset{i\text{-th}}{\frac{1}{P_1}} \otimes \cdots \otimes \mathbb{1}) \otimes (\mathbb{1} \otimes \cdots \otimes \underset{j\text{-th}}{P_1} \otimes \cdots \otimes \mathbb{1})$, acting on $\mathcal{H}^m \otimes \mathcal{H}^n$, $1 \leq i \leq m$ and $1 \leq j \leq n$.
- $\varphi_{i,j}^{m,n} := \varphi(P_{i,j}^{m,n})$ (functional calculus on $\mathcal{H}^m \otimes \mathcal{H}^n$).
- $\widetilde{R}_\varphi := \bigoplus_{m,n} \prod_{i,j} \varphi_{i,j}^{m,n}$

We can take the spectral decomposition of $\widetilde{R}_\varphi$ only with respect to the right component:

- $\widetilde{R}_\varphi = \bigoplus_n \int \prod_j \Gamma(\varphi(p_j P_1)) \otimes dE_1(p_1) \otimes \cdots \otimes dE_1(p_n)$

Note that the integrand is a unitary operator which implements a Longo-Witten endomorphism for any value of $p_j \geq 0$.

Massless construction

Input:

- $\mathcal{A}_0$: the net of the massive real free field
- P_+: the generator of positive-lightlike translation $T_0(a_+, 0) = e^{itP_0}$
- Ω_0: the vacuum
- $V(t) = e^{itP_0 \otimes P_0}, t \geq 0$

Interacting Borchers triple:

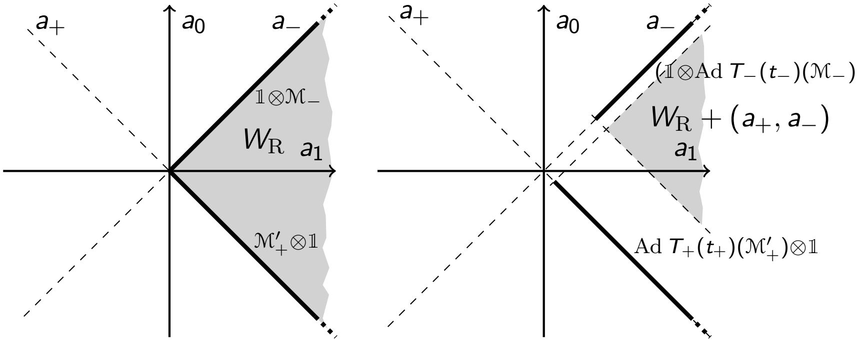
- $\mathcal{M}_t := (\mathcal{A}(W_\mathrm{R})' \otimes \mathbb{1}) \vee \mathrm{Ad}\, V(t)(\mathbb{1} \otimes \mathcal{A}(W_\mathrm{R}))$
- $T(a_+, a_-) := T_0(a_+) \otimes T_0(a_-)$
- $\Omega := \Omega_0 \otimes \Omega_0$

Theorem (T. arXiv:1107.2629)

$(\mathcal{M}_t, T, \Omega)$ is a massless Borchers triple with the S-matrix $V(t)$. Generalization possible for any inner symmetric function φ.

Matrix-valued φ and corresponding massive Borchers triples (Bischoff-T. arXiv:1305.2171)

Massless construction



Structure of massless net

Theorem (T. arXiv:1107.2629)

Let $(\mathcal{A}, T, \Omega)$ *be a massless* asymptotically complete *Haag-Kastler net (S-matrix* S *is defined on the whole Hilbert space* $\mathcal{H}$*) with standard properties (Bisognano-Wichmann property, Haag duality). Then there is a pair* $(\mathcal{A}_\pm, T_\pm, \Omega_\pm)$ *of (one-dimensional) conformal nets on* $\mathcal{H}_\pm$ *such that*

- $\mathcal{H} = \mathcal{H}_+ \otimes \mathcal{H}_-$
- $T(a_+, a_-) = T_+(a_+) \otimes T_-(a_-)$
- $\Omega = \Omega_+ \otimes \Omega_-$
- $\mathcal{A}(W_\mathrm{R}) = (\mathcal{A}_+(\mathbb{R}_-) \otimes \mathbb{1}) \vee \mathrm{Ad}S(\mathbb{1} \otimes \mathcal{A}_-(\mathbb{R}_+))$

Furthermore, for the modular objects of wedge and half-lines we have

- $\Delta = \Delta_+ \otimes \Delta_-$
- $J = S \cdot J_+ \otimes J_-$

Interacting massless net $=$ pair of conformal nets $+$ S-matrix.

Further examples

Use $\mathcal{A}_0 \subset \mathcal{F}$, where $\mathcal{F}$ is **the free complex fermion** net.
$\mathcal{A}_0$ is the fixed point with respect to the $U(1)$-gauge action.
The net $\mathcal{F} \otimes \mathcal{F}$ can be "twisted" by S_φ, and one can choose a twisting which commutes with the $U(1)$-gauge action, hence give rise to twisting of $\mathcal{A}_0 \otimes \mathcal{A}_0$.

The S-matrix **does not preserve** the subspace of one right-moving + one left-moving waves. In other words, they represent "particle production" (Bischoff-T. arXiv:1111.1671). Strict locality remains open.

cf. Massive models with "temperate" generators show no particle production (Borchers-Buchholz-Schroer '01)

Conclusion

Summary

- construction of interacting Haag-Kastler nets in 2 dimensions
- more Borchers triples, massive/massless
- structure of massless net: two conformal nets + S-matrix

Open problems

- more Borchers triples?
- strict locality of other examples?
- structure of massive/higher dimensional net?
- in higher dimensions?