Von Neumann algebras and ergodic theory of group actions

Mathematics and Quantum Physics, *in honor of Roberto Longo*

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KU LEUVEN

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$\rightsquigarrow$ As we shall see, the relation between the group Γ and its von Neumann algebra $L\Gamma$ is extremely subtle.

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$\rightsquigarrow$ (Voiculescu 1990, Radulescu 1993)
They are either all isomorphic, or all non-isomorphic.

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⤳ All icc groups Γ give us II_1 factors $L\Gamma$, but their structure is largely non-understood.

Amenable groups and Connes's theorem

Definition (von Neumann, 1929)

A group Γ is called **amenable** if there exists a finitely additive probability m on all the subsets of Γ that is translation invariant : $m(g\mathcal{U}) = m(\mathcal{U})$ for all $g \in \Gamma$ and $\mathcal{U} \subset \Gamma$.

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The following groups are amenable.

- Abelian groups, solvable groups.
- Stable under subgroups, extensions, direct limits.

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Conjecture (Connes, 1980)

Let Γ, Λ be icc groups with Kazhdan's property (T).
Then $L\Gamma \cong L\Lambda$ if and only if $\Gamma \cong \Lambda$.

W*-superrigidity for groups

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Theorem (Ioana-Popa-V, 2010)

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The same is true for $\mathcal{G} = (\mathbb{Z}/2\mathbb{Z})^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$,

for many groups Γ, including the free groups and arbitrary free product groups $\Gamma = \Gamma_1 * \Gamma_2$.

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- We will explain all these concepts, and some ideas behind the theorem.

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- The action $\mathrm{SL}(n, \mathbb{Z}) \curvearrowright \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$.
- The action $\Gamma \curvearrowright G/\Lambda$ for **lattices** $\Gamma, \Lambda < G$ in a Lie group G.

Group measure space construction (MvN, 1943)

Data : a pmp action $\Gamma \curvearrowright (X, \mu)$.

Output : von Neumann algebra $M = \mathrm{L}^\infty(X) \rtimes \Gamma$ with trace $\tau : M \to \mathbb{C}$.

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- Trace : $\tau(F) = \int F \, d\mu$ and $\tau(F \, u_g) = 0$ for $g \neq e$.

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⟿ If $\Gamma \curvearrowright (X, \mu)$ is free and ergodic, then $L^\infty(X) \rtimes \Gamma$ is a II_1 factor.

Actions of free groups

Theorem (Popa-V, 2011)

If $\mathbb{F}_n \curvearrowright X$ and $\mathbb{F}_m \curvearrowright Y$ are free ergodic pmp actions with $n \neq m$, then $\mathrm{L}^\infty(X) \rtimes \mathbb{F}_n \not\cong \mathrm{L}^\infty(Y) \rtimes \mathbb{F}_m$.

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If $\mathbb{F}_n \curvearrowright X$ and $\mathbb{F}_m \curvearrowright Y$ are free ergodic pmp actions with $n \neq m$, then $\mathrm{L}^\infty(X) \rtimes \mathbb{F}_n \not\cong \mathrm{L}^\infty(Y) \rtimes \mathbb{F}_m$.

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But $\mathrm{L}^\infty(X) \rtimes \mathbb{F}_\infty$ can have all kind of fundamental groups.

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  - up to unitary conjugacy : $A = uBu^*$ for some $u \in \mathcal{U}(M)$,
  - up to automorphic conjugacy : $A = \alpha(B)$ for some $\alpha \in \mathrm{Aut}(M)$.

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Speelman-V, 2011 : II_1 factors with **many** Cartan subalgebras (where "many" has a descriptive set theory meaning).

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Theorem (Popa-V, 2011)

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Profinite crossed products $L^\infty(X) \rtimes \mathbb{F}_n$ have a very special approximation property. $\rightsquigarrow$ Several conceptual novelties were needed to deal with non profinite actions.

Why uniqueness of Cartan subalgebras matters

Proposition (Singer, 1955). The following are equivalent.

- There exists an isomorphism $\pi : \mathrm{L}^\infty(X) \rtimes \Gamma \to \mathrm{L}^\infty(Y) \rtimes \Lambda$ with $\pi(\mathrm{L}^\infty(X)) = \mathrm{L}^\infty(Y)$.

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Corollary of our uniqueness of Cartan theorem : if $\mathrm{L}^\infty(X) \rtimes \mathbb{F}_n$ is isomorphic with $\mathrm{L}^\infty(Y) \rtimes \mathbb{F}_m$, then the actions $\mathbb{F}_n \curvearrowright X$ and $\mathbb{F}_m \curvearrowright Y$ must be orbit equivalent.

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Remaining question : do free groups $\mathbb{F}_n$, $\mathbb{F}_m$ with $n \neq m$ admit orbit equivalent actions ?

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Theorem (Gaboriau, 1999)

The cost of $\mathcal{R}(\mathbb{F}_n \curvearrowright X)$ is n. In particular, the free groups $\mathbb{F}_n$, $\mathbb{F}_m$ with $n \neq m$ do not admit orbit equivalent actions.

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- Direct products of the above groups.

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- Discrete subgroups of rank one simple Lie groups, like $SO(n, 1)$ or $SU(n, 1)$ or $Sp(n, 1)$.
- The limit groups of Sela.
- Direct products of the above groups.

Unique Cartan for **profinite** actions of the same groups : Chifan-Sinclair, 2011.

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Quite problematic : all known counterexamples to $\mathcal{C}$-rigidity admit an infinite amenable (almost) normal subgroup.