

(0)

# Free Probability with Left and Right Variables

(Free Probability for Pairs of Faces)

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Happy Birthday Roberto

(keep up the good fight

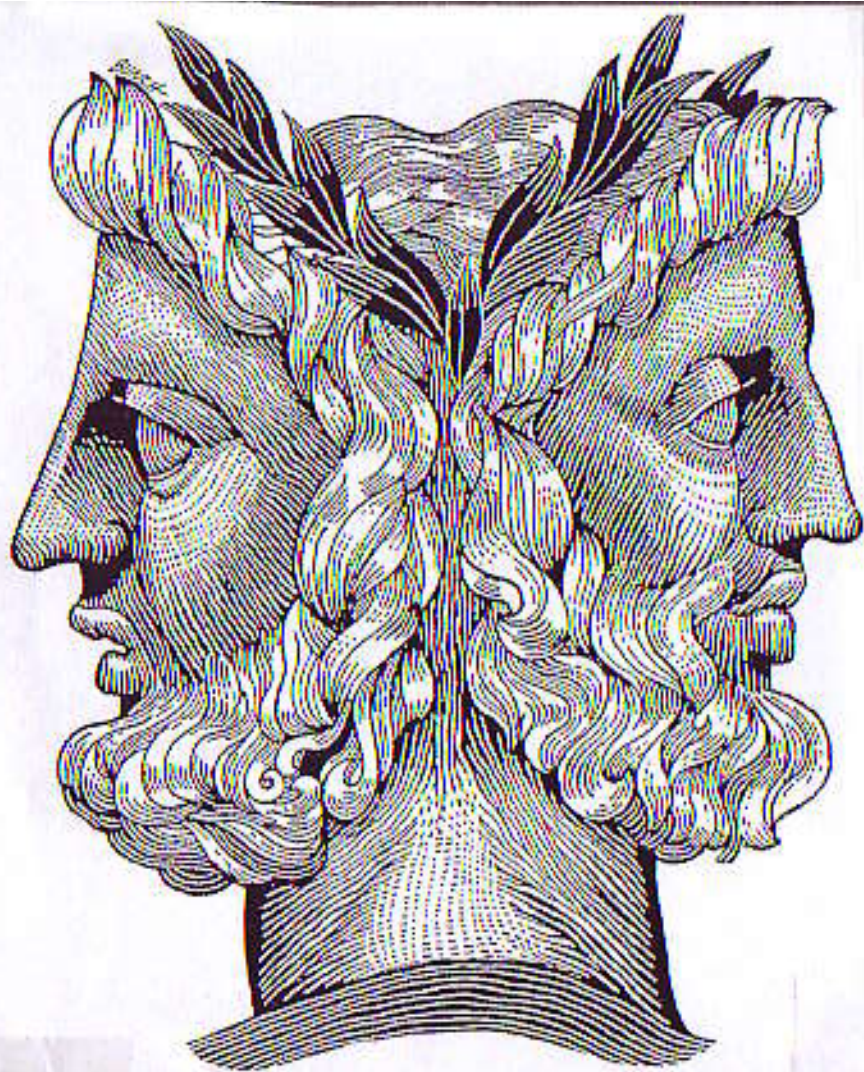
for von Neumann Algebras as

a Home for Quantum Fields)!

Free Probability with  
Left and Right Variables

(Free Probability for Pairs of Faces)

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Janus  
2 faces  
Past and Future  
Transition

[Left Var, Right Var] = 0

Bipartite  
System

## Possible Connections :

- Free Probability of Type B  
(Biane - Goodman - Nica,  
Belinschi - Shlyakhtenko)
- Second Order Freeness  
(Collins - Mingo - Sniady - Speicher)
- Matricial Freeness  
(Lenczewski)

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Free Product of Vector Spaces  
with specified State Vectors

$$\mathcal{X}_i = \overset{\circ}{\mathcal{X}}_i \oplus \mathbb{C} \xi_i$$

$$\mathcal{X} = \mathbb{C} \xi \oplus \underbrace{\bigoplus_{n \geq 1} \bigotimes_{i_1, \dots, i_n} \overset{\circ}{\mathcal{X}}_{i_1} \otimes \dots \otimes \overset{\circ}{\mathcal{X}}_{i_n}}_{\overset{\circ}{\mathcal{X}}}$$

$$(\mathcal{X}, \overset{\circ}{\mathcal{X}}, \xi) = \bigstar_{i \in I} (\mathcal{X}_i, \overset{\circ}{\mathcal{X}}_i, \xi_i)$$

$$\varphi_\xi: \mathcal{L}(\mathcal{X}) \rightarrow \mathbb{C}, \quad T \xi \in \varphi_\xi(T) \xi \oplus \overset{\circ}{\mathcal{X}}.$$

# Left and Right Factorizations (4)

$$V_c : \mathcal{X}_c \otimes \left( \mathbb{C} \xi \oplus \bigoplus_{m \geq 1} \bigotimes_{\substack{l_1 \neq l_2 \neq \dots \neq l_m \\ l_i \neq c}} \overset{\circ}{\mathcal{X}}_{l_1} \otimes \dots \otimes \overset{\circ}{\mathcal{X}}_{l_m} \right) \rightarrow \mathcal{X}$$

$$W_c : \left( \mathbb{C} \xi \oplus \bigoplus_{m \geq 1} \bigotimes_{\substack{l_1 \neq l_2 \neq \dots \neq l_m \\ l_i \neq c}} \overset{\circ}{\mathcal{X}}_{l_1} \otimes \dots \otimes \overset{\circ}{\mathcal{X}}_{l_m} \right) \otimes \mathcal{X}_c \rightarrow \mathcal{X}$$

$$T \in \mathcal{L}(\mathcal{X}_c)$$

$$\lambda_c(T) = V_c (T \otimes I) V_c^{-1} \in \mathcal{L}(\mathcal{X})$$

$$\rho_c(T) = W_c (I \otimes T) W_c^{-1} \in \mathcal{L}(\mathcal{X})$$

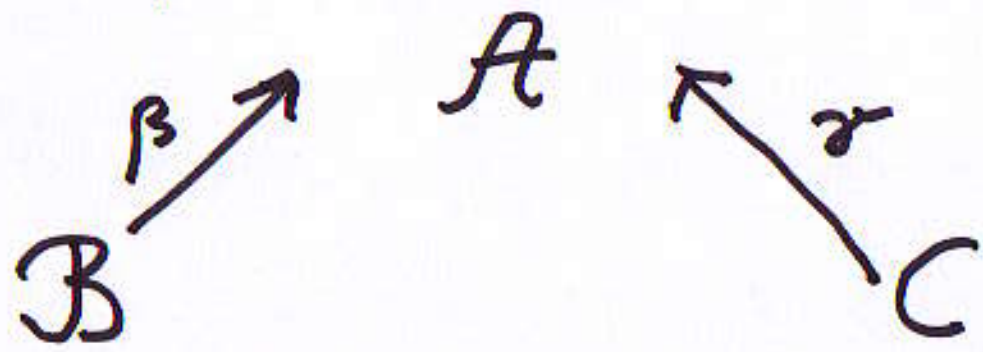
$$[\lambda_c(T), \rho_j(S)] = \delta_{ij} [T, S] \oplus 0.$$

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$(A, \varphi)$  noncommutative probability space

Pair of Faces in  $(A, \varphi)$

$(B, \beta)$  left face, right face  $(C, \gamma)$



$\beta, \gamma$  unital homomorphisms

$B, C$  unital algebras



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Included faces  $B \subset A \supset C$ .

( $\beta, \gamma$  are the inclusions)

2-faced family of noncommutative  
random variables in  $(A, \varphi)$

$((b_c)_{c \in I}, (c_j)_{j \in J})$  in  $A$

[Corresponds to

$$\beta: \mathbb{C}\langle X_c | c \in I \rangle \longrightarrow A, \quad \beta(X_c) = b_c$$

$$\gamma: \mathbb{C}\langle Y_j | j \in J \rangle \longrightarrow A, \quad \gamma(Y_j) = c_j$$

Bi-freeness of a family  
of pairs of faces

$((B_c, \beta_c), (C_c, \gamma_c))_{c \in I}$  in  $(A, \varphi)$ :

$\exists (\mathcal{X}_c, \mathring{\mathcal{X}}_c, \xi_c)_{c \in I}, (\mathcal{X}, \mathring{\mathcal{X}}, \xi) = \bigstar_{c \in I} (\mathcal{X}_c, \mathring{\mathcal{X}}_c, \xi_c)$

$l_c: B_c \rightarrow \mathcal{L}(\mathcal{X}_c), r_c: C_c \rightarrow \mathcal{L}(\mathcal{X}_c)$   
unital homomorphisms, so that

$$\varphi \circ \pi = \varphi_\xi \circ \tilde{\pi}$$

$$\pi: \bigstar_{c \in I} (B_c * C_c) \rightarrow A, \pi|_{B_c} = \beta_c, \pi|_{C_c} = \gamma_c$$

$$\tilde{\pi}: \bigstar_{c \in I} (B_c * C_c) \rightarrow \mathcal{L}(\mathcal{X}), \tilde{\pi}|_{B_c} = l_c \circ \lambda_c, \tilde{\pi}|_{C_c} = \xi_c \circ r_c$$

Remarks: 1° If  $((B_i, \beta_i), (C_i, \gamma_i))_{i \in I}$  bi-free (8)

in  $(A, \varphi)$ , joint distribution  $\varphi \circ \tilde{\pi}$  obtained also as  $\varphi_{\xi'} \circ \tilde{\pi}'$  for any other  $(\mathcal{X}', \mathcal{X}'_i, \xi')$ ,  $\ell', \pi'$  so that

$$\varphi \circ \tilde{\pi}_i = \varphi_{\xi'_i} \circ \tilde{\pi}'_i$$

$$\tilde{\pi}_i: B_i * C_i \rightarrow A, \quad \tilde{\pi}_i|_{B_i} = \beta_i, \quad \tilde{\pi}_i|_{C_i} = \gamma_i$$

$$\tilde{\pi}'_i: B_i * C_i \rightarrow \mathcal{L}(\mathcal{X}'_i), \quad \tilde{\pi}'_i|_{B_i} = \ell'_i, \quad \tilde{\pi}'_i|_{C_i} = \pi'_i$$

2°.  $\left( (B_i, \beta_i), (C_i, \gamma_i)_{i \in I} \right)$  bi-free in  $(A, \varphi)$   
 then  $(\beta_i(B_i))_{i \in I}$  free in  $(A, \varphi)$

$i \neq j \Rightarrow \beta_i(B_i), \gamma_j(C_j)$  classically independent in  $(A, \varphi)$ .

3°. bi-freeness has the necessary properties to be used as an independence relation in a noncommutative probability theory with left and right variables i.e. two-faced families.

4<sup>o</sup>.  $C^*$ -bi-freeness,  $W^*$ -bi-freeness  
 bi-free products of states etc.  
 bi-free convolution operations  
 (additive, multiplicative)

$$\mu \boxplus \boxplus \nu, \mu \boxtimes \boxtimes \nu$$

# Bi-freeness Examples

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I. Groups  $(G_i)_{i \in I}$ ,  $G = \ast_{i \in I} G_i$ .

$$L_i: \mathbb{C}[G_i] \rightarrow \mathcal{L}(\mathbb{C}[G])$$

$$R_i: \mathbb{C}[G_i]^{\text{op}} \rightarrow \mathcal{L}(\mathbb{C}[G])$$

restrictions of left and right regular representations

$$((\mathbb{C}[G_i], L_i), (\mathbb{C}[G_i]^{\text{op}}, R_i))_{i \in I}$$

bi-free family of faces in  $(\mathbb{C}[G], \mathcal{L})$ .  
v. Neumann trace

II. Left and right creation and annihilation operators on the full Fock space.

$\mathcal{H}$  complex Hilbert sp.  $(e_i)_{i \in I}$  ONB

$$\mathcal{T}(\mathcal{H}) = \mathbb{C}1 \oplus \bigoplus_{n \geq 1} \mathcal{H}^{\otimes n}$$

$$l_i \zeta = e_i \otimes \zeta, \quad r_i \zeta = \zeta \otimes e_i, \quad \zeta \in \mathcal{T}(\mathcal{H}).$$

$$\omega(T) = \langle T1, 1 \rangle \text{ on } \mathcal{B}(\mathcal{T}(\mathcal{H}))$$

$$\left( (l_i, l_i^*), (r_i, r_i^*) \right)_{i \in I}$$

bi-free in  $(\mathcal{B}(\mathcal{T}(\mathcal{H}), \omega)$ .

# Bi-free Cumulants

$z = ((z_i)_{i \in I}, (z_j)_{j \in J})$  2-faced family of n.v. in  $(A, \varphi)$

Moments  $\varphi(z_{\alpha(1)} \cdots z_{\alpha(n)})$ ,  $\alpha: \{1, \dots, n\} \rightarrow I \sqcup J$

$R_\alpha$  polynomial in commuting variables  $X_{\alpha(k_1) \cdots \alpha(k_n)}$ ,  $1 \leq k_1 < \dots < k_n \leq n$ .

homogeneous  $\deg = n$ ,  $\deg X_{\alpha(k_1) \cdots \alpha(k_n)} = n$ .



$$R_\alpha(z) = R_\alpha(\varphi(z_{\alpha(k_1)}, \dots, z_{\alpha(k_n)}) \mid 1 \leq k_1 < \dots < k_n \leq n) \quad (14)$$

$R_\alpha$  bi-free cumulant, exists & unique  
so that:

1<sup>o</sup>. coefficient of  $X_{\alpha(1)} \dots X_{\alpha(n)} = 1$

2<sup>o</sup>.  $z', z''$  bi-free in  $(A, \varphi)$ , then

$$R_\alpha(z') + R_\alpha(z'') = R_\alpha(z' + z'').$$

$$\alpha \Pi_m = \{ (\alpha(k_1), \dots, \alpha(k_n)) \mid 1 \leq k_1 < \dots < k_n \leq n, 1 \leq n \leq m \} \quad (15)$$

$$M_{z, \alpha} = \left( \varphi(z_{\alpha(i)} - z_{\alpha(k_n)}) \right)_{(\alpha(k_1), \dots, \alpha(k_n)) \in \alpha \Pi_m}$$

$$(M_{z', \alpha}, M_{z'', \alpha}) \longrightarrow M_{z' + z'', \alpha}$$

polynomial abelian group law on  $\mathbb{C}^{\alpha \Pi_m}$

$\mathbb{C}^{\alpha \Pi_m} \xrightarrow{\exp} \mathbb{C}^{\alpha \Pi_m}$  isomorphism  
 (Lie algebra, +)  $\boxplus$   $\boxplus_m$  law

$\log = (\exp)^{-1}$  yields cumulants

# Bi-free Central Limit

$\mathfrak{z}$  two-faced family in  $(A, \varphi)$   
 has bi-free central limit distribution  
 (aka bi-free Gaussian)

if  $n \neq 2 \implies R_{\alpha(1) \dots \alpha(n)}(\mathfrak{z}) = 0$

$n = 1 \quad R_a(\mathfrak{z}) = \varphi(z_a)$

$n = 2 \quad R_{ab}(\mathfrak{z}) = \varphi(z_a z_b) - \varphi(z_a)\varphi(z_b).$

bi-free central limit distribution

$$\gamma_c: \mathbb{C}\langle Z_k \mid k \in I \sqcup J \rangle \rightarrow \mathbb{C}$$

determined by covariance matrix

$$C = (C_{k,e})_{k,e \in I \sqcup J}$$

$$\gamma_c(Z_k Z_e) = C_{ke}$$

$$(\text{equivalently } C_{ke} = R_{ke}(Z)).$$

# Realization on full Fock space

$\mathcal{F}(\mathcal{H})$  full Fock space,  $T \rightarrow \langle T1, 1 \rangle$   
vacuum expectation

$l(h), l^*(h), r(h), r^*(h)$

left and right creation and annihilation

$h, h^*: I \amalg J \rightarrow \mathcal{H}$  maps

$$z_i = l(h(i)) + l^*(h^*(i)) \quad i \in I$$

$$z_j = r(h(j)) + r^*(h^*(j)) \quad j \in J$$

$z = ((z_i)_{i \in I}, (z_j)_{j \in J})$  bi-free Gaussian

covariance  $C_{ab} = \langle h(b), h^*(a) \rangle$ .

# Bi-free Algebraic CLT

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bi-free sequence

$$(z^{(n)})_{n \in \mathbb{N}} = \left( (z_i^{(n)})_{i \in I}, (z_j^{(n)})_{j \in J} \right)_{n \in \mathbb{N}} \text{ in } (A, \varphi)$$

(i)  $\varphi(z_h^{(n)}) = 0$ ,  $h \in I \cup J$

(ii)  $\sup_{n \in \mathbb{N}} |\varphi(z_{k_1}^{(n)} \cdots z_{k_m}^{(n)})| = D_{k_1, \dots, k_m} < \infty$

(iii)  $\lim_{N \rightarrow \infty} N^{-1} \sum_{1 \leq n \leq N} \varphi(z_h^{(n)} z_l^{(n)}) = C_{h,l}$

$$S_N = \left( (S_{N,i})_{i \in I}, (S_{N,j})_{j \in J} \right)$$

$$S_{N,h} = N^{-1/2} \sum_{1 \leq n \leq N} z_h^{(n)}$$

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$\Rightarrow S_N$  has limit distribution bi-free Gaussian with covariance  $(C_{h,l})_{h,l \in I \cup J}$  as  $N \rightarrow \infty$

$\mathbb{C} \rightsquigarrow \mathcal{B}$  algebra with 1.

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Bi-freeness with amalgamation  
over  $\mathcal{B}$

$\mathcal{B}$ - $\mathcal{B}$  noncommutative probability space

$(A, \rho, \varepsilon)$   $A$  unital algebra over  $\mathbb{C}$

$\varepsilon: \mathcal{B} \otimes \mathcal{B}^{\text{op}} \rightarrow A$  unital homomorphism

$\varepsilon|_{\mathcal{B} \otimes 1}, \varepsilon|_{1 \otimes \mathcal{B}^{\text{op}}}$  injective

$\rho: A \rightarrow \mathcal{B}$  linear unital

$\rho(\varepsilon(b_1 \otimes 1) a \varepsilon(1 \otimes b_2)) = b_1 \rho(a) b_2$

(in particular  $(\rho \circ \varepsilon)(b_1 \otimes b_2) = b_1 b_2$  .

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 $(A, \rho, \varepsilon)$   $B$ - $B$  noncommutative probability space

$A_n$  commutant in  $A$  of  $\varepsilon(1 \otimes B^{op})$

$A_e$  commutant in  $A$  of  $\varepsilon(B \otimes 1)$

included pair of  $B$ -faces in  $(A, \rho, \varepsilon)$

$(C, D)$  unital subalgebras in  $A$

$$\varepsilon(B \otimes 1) \subset C \subset A_n$$

$$\varepsilon(1 \otimes B^{op}) \subset D \subset A_e$$



$\mathcal{B}$ - $\mathcal{B}$  bimodules with specified  
state vector

$$\mathcal{X} = \overset{\circ}{\mathcal{X}} \oplus \mathcal{B} \quad \mathcal{X}, \overset{\circ}{\mathcal{X}} \text{ } \mathcal{B}\text{-}\mathcal{B} \text{ bimodules}$$

Free Product

$$\bigstar_{\mathcal{B}}^{\mathcal{I}} (\mathcal{X}_{l_i}, \overset{\circ}{\mathcal{X}}_i) = (\mathcal{X}, \overset{\circ}{\mathcal{X}})$$

$$\overset{\circ}{\mathcal{X}} = \bigoplus_{n \geq 1} \bigoplus_{l_1 \neq \dots \neq l_n} \mathcal{X}_{l_1} \otimes_{\mathcal{B}} \mathcal{X}_{l_2} \otimes_{\mathcal{B}} \dots \otimes_{\mathcal{B}} \mathcal{X}_{l_n}$$

$$\mathcal{X} = \overset{\circ}{\mathcal{X}} \oplus \mathcal{B}$$

$\mathcal{X} = \overset{\circ}{\mathcal{X}} \oplus \mathcal{B}$   $\mathcal{B}$ - $\mathcal{B}$  bimodule

$p_{\mathcal{X}} : \mathcal{L}(\mathcal{X}) \rightarrow \mathcal{B}$

$\tau(0 \oplus 1) \in \overset{\circ}{\mathcal{X}} \oplus p_{\mathcal{X}}(\tau)$

$\varepsilon_{\mathcal{X}} : \mathcal{B} \oplus \mathcal{B}^{\text{op}} \rightarrow \mathcal{L}(\mathcal{X})$  left & right  $\mathcal{B}$  multipliers

$(\mathcal{L}(\mathcal{X}), p_{\mathcal{X}}, \varepsilon_{\mathcal{X}})$   $\mathcal{B}$ - $\mathcal{B}$  noncommutative probability space

$\mathcal{L}_r(\mathcal{X})$  right  $\mathcal{B}$ -linear operators

$\mathcal{L}_l(\mathcal{X})$  left  $\mathcal{B}$ -linear operators

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$$(\mathcal{X}, \overset{\circ}{\mathcal{X}}) = \bigstar_{l \in I} \mathcal{B} (\mathcal{X}_l, \overset{\circ}{\mathcal{X}}_l)$$

$$V_l: \mathcal{X}_l \otimes_{\mathcal{B}} \left( \mathcal{B} \oplus \bigoplus_{m \geq 1} \bigoplus_{l_1 \neq l_2 \neq \dots \neq l_m} \overset{\circ}{\mathcal{X}}_{l_1} \otimes_{\mathcal{B}} \dots \otimes_{\mathcal{B}} \overset{\circ}{\mathcal{X}}_{l_m} \right) \rightarrow \mathcal{X}$$

$$W_l: \left( \mathcal{B} \oplus \bigoplus_{m \geq 1} \bigoplus_{l_1 \neq \dots \neq l_m \neq l} \overset{\circ}{\mathcal{X}}_{l_1} \otimes_{\mathcal{B}} \dots \otimes_{\mathcal{B}} \overset{\circ}{\mathcal{X}}_{l_m} \right) \otimes \mathcal{X}_l \rightarrow \mathcal{X}$$

$$\lambda_l: \mathcal{L}_r(\mathcal{X}_l) \rightarrow \mathcal{L}_r(\mathcal{X})$$

$$\rho_l: \mathcal{L}_l(\mathcal{X}_l) \rightarrow \mathcal{L}_l(\mathcal{X})$$

$$\lambda_l(T) = V_l (T \otimes I) V_l^{-1}$$

$$\rho_l(T) = W_l (I \otimes T) W_l^{-1}$$

$(A, p, \varepsilon)$   $B$ - $B$  noncomm. probs. sp.

family  $((C_i, D_i))_{i \in I}$  of pairs of  $B$ -face

in  $(A, p, \varepsilon)$  is bi-free over  $B$  if:

$\exists \mathcal{X}_i = \mathring{\mathcal{X}}_i \oplus B$   $B$ - $B$  bimodules  
unital homomorphisms

$$\gamma_i : C_i \rightarrow \mathcal{L}_n(\mathcal{X}_i), \quad \gamma_i(\varepsilon(b \otimes 1)) = \varepsilon_{\mathcal{X}_i}(b \otimes 1)$$

$$\delta_i : D_i \rightarrow \mathcal{L}_l(\mathcal{X}_i), \quad \delta_i(\varepsilon(1 \otimes b)) = \varepsilon_{\mathcal{X}_i}(1 \otimes b)$$

so that

if  $c_k \in C_{L(k)}, d_k \in D_{L(k)}, 1 \leq k \leq n$  (26)

then

$$p(c_1 d_1 c_2 d_2 \dots c_n d_n) =$$

$$= p_{\mathcal{X}}(\lambda_{i(1)}(\gamma_{L(1)}(c_1)) \rho_{L(1)}(\delta_{L(1)}(d_1)) \dots$$

$$\dots \lambda_{i(n)}(\gamma_{L(n)}(c_n)) \rho_{L(n)}(\delta_{L(n)}(d_n))).$$

