Fluctuations of random tilings and discrete Beta-ensembles

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Joint work with A. Borodin, G. Borot, V. Gorin, J. Huang
Consider an hexagon with a hole and take a tiling at random. How does it look?
When the mesh of the tiling goes to zero, one can see a “frozen” region and a “liquid” region. Limits, fluctuations?
Cohn, Larsen, Propp 98': When tiling an hexagon, the shape of the tiling converges almost surely as the mesh goes to zero.
General domains

Cohn-Kenyon-Propp 00' and Kenyon-Okounkov 07': The shape of the tiling (e.g., the height function) converges almost surely for a large class of domains.
Fluctuations of the surface

**Conjecture.** (Kenyon-Okounkov) The recentered height function converges to the **Gaussian Free Field** in the liquid region in general domains.

- (Kenyon-06’) A class of domains with no frozen regions
- (Borodin–Ferrari-08’) Some *infinite* domains with frozen regions
- (Boutillier-de Tilière-09’, Dubedat-11’) On the torus
- (Petrov-12’, Bufetov-G.-16’) A class of simply-connected polygons
- (Berestycki-Laslier-Ray-16+) *Flat* domains, some manifolds
- Borodin-Gorin-G.- 16’, and Bufetov-Gorin.-17’ Polygons with holes — *trapezoid gluings.*
Local fluctuations of the boundary of the liquid region

Ferrari-Spohn 02’, Baik-Kriecherbauer-McLaughlin-Miller 03’: appropriately rescaled, a generic point in the boundary of the liquid region converges to the Tracy-Widom distribution in the random tiling of the hexagon, the distribution of the fluctuations of the largest eigenvalue of the GUE.

G-Huang. -17’: This extends to polygonal domains obtained by trapezoid gluings (on the gluing axis).
Trapezoids gluings

We can glue arbitrary many trapezoids, where we may cut triangles or lines, always along a single vertical axis.
Trapezoids gluings

- We can glue arbitrary many trapezoids, where we may cut triangles or lines,
- Always along a **single** vertical axis
What is good about trapezoids?

**Fact:** The total number of tilings of trapezoid with **fixed** along the border horizontal lozenges $\ell_N > \cdots > \ell_1$ is proportional to

$$\prod_{i<j} \frac{\ell_j - \ell_i}{j - i}$$

Indeed, Tilings $\approx$ Gelfand–Tsetlin patterns, enumerated through combinatorics of Schur polynomials or characters of unitary groups.
Distribution of horizontal tiles

$H = 4$ cuts

The distribution of horizontal lozenges $\{\ell^h_i\}$ along the axis of gluing has the form: $\ell^h_{i+1} \geq \ell_i + 1$

$$P_{N}^{\Theta,w}(\ell) = \frac{1}{Z_N^{\Theta,w}} \prod_{i<j}(\ell_i - \ell_j)^{2\Theta[h(i),h(j)]} \prod_{i=1}^{N} w(\ell_i)$$

$h(i)$ — number of the cut. $\Theta$ — symmetric $H \times H$ matrix of 1’s, 1/2’s, and 0’s with 1’s on the diagonal.
Discrete $\beta$-ensembles ($\beta = 2\theta$)

For configurations $\ell$ such that $\ell_{i+1}^h - \ell_i^h - \theta_{h,h} \in \mathbb{N}$, it is given by:

$$P_{N}^{\theta,w}(\ell) = \frac{1}{Z_N} \prod_{1 \leq h \leq h' \leq H} \prod_{1 \leq i \leq N_h} l_{\theta_{h,h'}}(\ell_{j}^{h'}, \ell_{i}^{h}) \prod_{1 \leq j \leq N_{h}, i < j} w(\ell_{i}^{h}),$$

where $l_{\theta}(\ell', \ell) = \frac{\Gamma(\ell' - \ell + 1)\Gamma(\ell' - \ell + \theta)}{\Gamma(\ell' - \ell)\Gamma(\ell' - \ell + 1 - \theta)}$

Note that $l_{\theta}(\ell', \ell) \sim |\ell' - \ell|^{2\theta}$ with $\sim$ if $\theta = 1, 1/2$. 
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\]

where $I_{\theta}(\ell', \ell) = \frac{\Gamma(\ell' - \ell + 1)\Gamma(\ell' - \ell + \theta)\Gamma(\ell' - \ell')\Gamma(\ell' - \ell + 1 - \theta)}{\Gamma(\ell' - \ell)\Gamma(\ell' - \ell + \theta)\Gamma(\ell' - \ell')\Gamma(\ell' - \ell + 1 - \theta)}$

Note that $I_{\theta}(\ell', \ell) \simeq |\ell' - \ell|^{2\theta}$ with $= \text{if } \theta = 1, 1/2$.

- We can study the convergence, global fluctuations of the empirical measures

\[
\hat{\mu}_N^h = \frac{1}{N_h} \sum_{i=1}^{N_h} \delta_{\ell_i^h/N}, 1 \leq h \leq H
\]

and fluctuations of the extreme particles.
Discrete $\beta$-ensembles ($\beta = 2\theta$)

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where $I_{\theta}(\ell', \ell) = \frac{\Gamma(\ell' - \ell + 1)\Gamma(\ell' - \ell + \theta)}{\Gamma(\ell' - \ell)\Gamma(\ell' - \ell + 1 - \theta)}$

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and fluctuations of the extreme particles.

- Bufetov-Gorin 17': the fluctuations of the surface of the whole tiling follows from the fluctuations on the gluing axis.
Discrete $\beta$-ensembles ($\beta = 2\theta$): law of large numbers

Assume $w(x) \simeq e^{-NV(x/N)}$, $(\theta_{h,h'})_{h,h'} \geq 0$, $\theta_{h,h} > 0$.

- **Fixed heights**: $N_h/N \mapsto \varepsilon_h$, Then

$$
\lim_{N \to \infty} \frac{1}{N_h} \sum_{i=1}^{N_h} \delta_{\ell_i^h/N} \to \mu_{\varepsilon}^h \ a.s.,
$$

- **Random heights**: $\sum_h N_h = N$, Then $N_h/N \to \varepsilon_*^h$ and

$$
\lim_{N \to \infty} \frac{1}{N_h} \sum_{i=1}^{N_h} \delta_{\ell_i^h/N} \to \mu_{\varepsilon_*}^h \ a.s.,
$$

Indeed,

$$
P_{N}^{\theta,w}(\ell) \simeq e^{-N^2\varepsilon(\frac{1}{N_h} \sum_{i=1}^{N_h} \delta_{\ell_i^h/N} , 1 \leq h \leq H)}
$$

where $\varepsilon$ has a unique minimizer.
Assumption on the equilibrium measures towards fluctuations

Note that for all \( h \):

\[
0 \leq \frac{d\mu_h^\varepsilon}{dx} \leq \theta_{hh}^{-1}
\]

We shall assume

- The liquid region \( \{ 0 < \frac{d\mu_h^\varepsilon}{dx} < \theta_{hh}^{-1} \} \) are connected,
- The equilibrium measures are off critical: at the boundary of the liquid region they behave like a square root.

\[
\frac{w(x)}{w(x-1)} = \frac{\phi^+_N(x)}{\phi^-_N(x)}, \quad \phi^\pm_N \text{ analytic }, \phi^\pm = \phi^\pm + \frac{1}{N}\phi^\pm_1 + o\left(\frac{1}{N}\right)
\]

Rmk: Off-criticality should be generically true.
Global fluctuations: fixed heights

Assume $N_h/N \mapsto \varepsilon_h$.

Theorem (Borodin-Gorin-G 15’ Borot-Gorin-G 17’)

Then for any analytic functions $f_h$:

$$\left( \sum_{i=1}^{N_h} (f_h(\ell^h_i/N) - \mathbb{E}[f_h(\ell^h_i/N)]) \right)_h \Rightarrow N(0, \Sigma(f)).$$
Global fluctuations: Random Heights

Assume $\sum N_i = N$. Then [WIP Borot-Gorin-G ]

\[
\frac{N_i}{N} \to \varepsilon_i^*
\]

The heights are equivalent to discrete Gaussian ‘:

\[
P_{N}^{\theta,w}(N_h - \mathbb{E}[N_h] = x) \approx \frac{1}{Z} e^{-\frac{1}{2\sigma}(x)^2}
\]
Global fluctuations: Random Heights

Assume $\sum N_i = N$. Then [WIP Borot-Gorin-G ]

$\frac{N_i}{N} \to \varepsilon^*_i$

The heights are equivalent to discrete Gaussian ‘:

$$P_{\theta, w}^N (N_h - \mathbb{E}[N_h] = x) \sim \frac{1}{Z} e^{-\frac{1}{2\sigma}(x)^2}$$

$$\sum_{i=1}^{N_h} (f_h(\ell_i^h / N) - \mathbb{E}[f_h(\ell_i^h / N)]) - \sum_k (N_k - \mathbb{E}[N_k]) \frac{\partial}{\partial \varepsilon} \mu^h_{\varepsilon}(f_h)|_{\varepsilon = \varepsilon^*}$$

converges towards a centered Gaussian variable.
Under the previous assumptions, the boundary fluctuates like a Tracy-Widom distribution.

If \( \frac{1}{N_1} \sum \delta_{\ell^1_i/N} \) converges towards \( \mu^1 \) with liquid region \([a, b]\), \( \mu^1((-\infty, a)) = 0 \), then for all \( t \) real

\[
\lim_{N \to \infty} P_{\theta}^{\theta,w} \left( N^{2/3} (\ell^1_1/N - a) \geq t \right) = f_{2\theta_{11}}(t)
\]

with \( f_{2\theta} \) the \( 2\theta \)-Tracy-Widom distribution appearing in the **continuous** \( \beta \)-ensembles.

**Corollary:** Fluctuations of the first rows of Young diagrams under Jack deformation of Plancherel measure.
Continuous $\beta$-ensembles

The distribution of continuous $\beta$-ensembles is given by

$$ dP_{N}^{\beta,V}(\lambda) = \frac{1}{\tilde{Z}_{N}^{\beta,V}} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-\beta N \sum_{i=1}^{N} V(\lambda_i)} \prod d\lambda_i. $$

- When $V(x) = \frac{1}{4} x^2$, and $\beta = 1$ (resp. $\beta = 2, 4$), $P_{N}^{\beta,V}$ is the distribution of the eigenvalues of a symmetric (resp. Hermitian, resp. symplectic) $N \times N$ matrix with centered Gaussian entries with covariance $1/N$, the GOE (resp. GUE, resp. GSE).
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- when $w(x) \sim e^{-\beta N V(x/N)}$,

$$Z_{N}^{\theta,w} P_{N}^{\theta,w}(\ell) \sim \tilde{Z}_{N}^{2\theta,V} dP_{N}^{2\theta,V}(\ell/N)$$
Some techniques to deal with $\beta$-ensembles

- **Integrable systems.** In some cases, these distributions have particular symmetries allowing for special analysis, e.g., when $\beta = 2\theta = 2$ the density is the square of a determinant, allowing for orthogonal polynomials analysis [Mehta, Deift, Baik, Johansson etc]

- **General case.**
  - **Dyson-Schwinger (or loop) Equations.** Use equations for the correlators, e.g., the moments of the empirical measures, and try to solve them asymptotically [Johansson, Shcherbina, Borot-G, etc]
  - **Universality.** Compare your law to one you can analyze [Erdös-Yau et al, Tao-Vu]. An important step is to prove *rigidity* (particles are very close to their deterministic limit), see [Bourgade-Erdös-Yau]
Continuous $\beta$-ensembles: Dyson-Schwinger equations (DSE)

Let
\[ \hat{\mu}^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i} : \hat{\mu}_N(f) = \frac{1}{N} \sum_{i=1}^{N} f(\lambda_i) \]

If $f$ is a smooth test function, then integration by parts yields the DSE:
\[ N \mathbb{E} \left[ \hat{\mu}_N(V' f) - \frac{1}{2} \int \frac{f(x) - f(y)}{x - y} d\hat{\mu}_N(x) d\hat{\mu}_N(y) \right] = \left( \frac{1}{\beta} - \frac{1}{2} \right) \mathbb{E} [\hat{\mu}_N(f'_0)] \]
Continuous $\beta$-ensembles: Dyson-Schwinger equations

Linearizing this equation around its limit, we get

$$N\mathbb{E}[(\hat{\mu}_N - \mu)(\Xi f)] = \left(\frac{1}{\beta} - \frac{1}{2}\right)\mathbb{E}[\hat{\mu}_N(f')]$$

$$+ \frac{1}{2N}\mathbb{E}\left[\int \int \frac{f(x) - f(y)}{x - y} dN(\hat{\mu}_N - \mu)(x)dN(\hat{\mu}_N - \mu)(y)\right]$$

where

$$\Xi f(x) = V'(x)f(x) - \int \frac{f(x) - f(y)}{x - y} d\mu(y).$$

If we can show that the last term is negligible and $\Xi$ is invertible (off-criticality), we can solve asymptotically this equation to get

$$N\mathbb{E}[\hat{\mu}_N - \mu](f) \simeq \left(\frac{1}{\beta} - \frac{1}{2}\right)\mathbb{E}[\hat{\mu}_N((\Xi^{-1}f)')]$$

We can get an infinite system of DSE for all moments of $\hat{\mu}_N$ and get large $N$ expansion up to any order. This gives the CLT.
Continuous $\beta$-ensembles: Fluctuations at the edge

Dumitriu-Edelman 02': Take $V(x) = \beta x^2 / 2$. Then $P_N^{\beta, \beta x^2 / 2}$ is the law of the eigenvalues of

$$H_N^{\beta} = \begin{pmatrix}
Y_1^{\beta} & \xi_1 & 0 & \cdots & 0 \\
\xi_1 & Y_2^{\beta} & \xi_2 & 0 & \vdots \\
0 & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \xi_{N-1} & Y_N^{\beta}
\end{pmatrix}$$

where $\xi_i$ are iid $N(0, 1)$ and $Y_i^{\beta} \sim \chi_i \beta$ independent.
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0 & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \xi_{N-1} & Y_N^\beta
\end{pmatrix}$$

where $\xi_i$ are iid $N(0,1)$ and $Y_i^\beta \sim \chi_i\beta$ independent.

Ramirez-Rider-Virág 06': The largest eigenvalue fluctuates like Tracy-Widom $\beta$ distribution.

Bourgade-Erdős-Yau 11’, Shcherbina 13’, Bekerman-Figalli-G 13’: Universality: This remains true for general potentials provided off-criticality holds.
Discrete $\beta$-ensembles: Nekrasov’s equation

Recall $\ell_{i+1} - \ell_i - \theta \in \mathbb{N}$ and set

$$P_{\theta,w}^N(\ell) = \frac{1}{Z_N} \prod_{1 \leq i < j \leq N} \frac{\Gamma(\ell_j - \ell_i + 1)\Gamma(\ell_j - \ell_i + \theta)}{\Gamma(\ell_j - \ell_i)\Gamma(\ell_j - \ell_i + 1 - \theta)} \prod w(\ell_i, N)$$

and assume there exists $\phi^\pm_N$ analytic so that

$$\frac{w(x, N)}{w(x - 1, N)} = \frac{\phi^+_N(x)}{\phi^-_N(x)}.$$

Then

$$\phi^-_N(\xi) \mathbb{E}_{P_{\theta,w}^N} \left[ \prod_{i=1}^N \left( 1 - \frac{\theta}{\xi - \ell_i} \right) \right] + \phi^+_N(\xi) \mathbb{E}_{P_{\theta,w}^N} \left[ \prod_{i=1}^N \left( 1 + \frac{\theta}{\xi - \ell_i - 1} \right) \right]$$

is analytic.
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is analytic.

$$\Rightarrow \text{ Gives asymptotic equations for } G_{N}(z) := \frac{1}{N} \sum_{i=1}^{N} \frac{1}{z - \ell_i/N}$$

which can be analyzed as DSE.
Consequences of Nekrasov’s equation

- One can estimate all moments of $N(G_N(z) - G(z))$ for $z \in \mathbb{C} \setminus \mathbb{R}$, proving CLT,
Consequences of Nekrasov’s equation

- One can estimate all moments of $N(G_N(z) - G(z))$ for $z \in \mathbb{C} \setminus \mathbb{R}$, proving CLT,

- One can estimate all moments of $N(G_N(z) - G(z))$ for $\Im z = \frac{1}{N^{1-\delta}}$ proving rigidity: for any $a > 0$

$$P_{N}^{\theta,w}(\sup_{i} |\ell_i - N\gamma_i| \geq \frac{N^a}{\min\{i/N, 1 - i/N\}^{1/3}}) \leq e^{-(\log N)^2}$$

where $\mu((\infty, \gamma_i)) = i/N$. 
Consequences of Nekrasov’s equation

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- One can estimate all moments of $N(G_N(z) - G(z))$ for \( \Im z = \frac{1}{N^{1-\delta}} \) proving rigidity: for any $a > 0$

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\]

where $\mu((\infty, \gamma_i)) = i/N$.

- One can compare the law of the extreme particles, at distance of order $N^{1/3} \gg 1$ (the mesh of the tiling) with the law of the extreme particles for the continuous model and deduce the $2\theta$-Tracy-Widom fluctuations.
Some open questions

- **Critical case.** In continuous setting, Bekerman, Leblé, Serfaty 17’ derived CLT for functions in \( \text{Im}(\Xi) \). What about the discrete case? Can we get universality of local fluctuations?
- **Universality for CLT.** What if we have true Coulomb gas interaction in the discrete case?
- **General domains?**
- **Local fluctuations in the bulk.** When \( \theta = 1 \) correlations functions in the bulk converge to discrete Sine process. What about other \( \theta \)?
Many thanks to Alexei Borodin, Vadim Gorin and Leonid Petrov for the pictures.
And thank you for your attention!