Symmetries in Quantum Field Theory beyond Groups *

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Quantum Theory

Observables: $*$-algebra, e.g. von Neumann algebra $M \subset \mathrm{B}(\mathcal{H}_M)$
States: $\omega_M \colon M \to \mathbb{C}$ positive normalized linear functional

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The right maps between (M, ω_M) and (N, ω_N) are $\phi\colon M \to N$, such that

1. **linear** $\phi(m + \lambda m') = \phi(m) + \lambda\phi(m')$
2. **unital** $\phi(1_M) = 1_N$
3. **completely positive**, i.e. the amplified maps

$$\phi_n = \phi \otimes \mathrm{id}\colon M \otimes \mathrm{Mat}_n(\mathbb{C}) \to N \otimes \mathrm{Mat}_n(\mathbb{C})$$

   are positive.
   - E.g. $*$-homomorphisms $\rho\colon M \to N$
   - $\exists$ a representation $\pi\colon M \to \mathrm{B}(\mathcal{K})$ and an isometry $V\colon \mathcal{H}_N \to \mathcal{K}$, such that $\phi(\,\cdot\,) = V^*\pi(\,\cdot\,)V$. In particular, $\phi(m^*) = \phi(m)^*$
4. **stochastic** $\omega_N(\phi(m)) = \omega_M(m)$ for all $m \in M$.

Global Symmetries in QFT

Quantum Field Theory

Idea:

$$\mathcal{A}(O) = \{\phi(f)\colon \operatorname{supp}(f) \subset O\}'' \qquad \mathcal{H} = \overline{\cup_O \mathcal{A}(O)}\Omega$$

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Abstractly, net of von Neumann algebras: $(O \mapsto \mathcal{A}(O), \mathcal{H}, \Omega \in \mathcal{H})$

- $O_1 \subset O_2 \Rightarrow \mathcal{A}(O_1) \subset \mathcal{A}(O_2)$
- O_1, O_2 spacelike separated $\Rightarrow \mathcal{A}(O_1)$ an $\mathcal{A}(O_2)$ commute
- $\omega(\,\cdot\,) = (\Omega, \cdot\,\Omega)$ vacuum state.

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Remark

We get a unitary U_α by $U_\alpha m\Omega = \alpha_O(m)\Omega$, which implements α, i.e. $\alpha_O(\,\cdot\,) = U_\alpha \cdot U_\alpha^*$ for all O and U_α commutes with space-time symmetries.

If $\mathcal{A}$ is a nice QFT net of observables, then

- $\exists!\ (G,k)$ with G a compact group and $k \in Z(G)$ and involution
- $\exists!\ \mathcal{F} \supset \mathcal{A}$ a (possible $\mathbb{Z}_2$-graded) local field net with $\mathrm{DHR}(\mathcal{F})$ trivial and $G \leq \mathrm{Aut}(\mathcal{F})$, such that

$$\mathcal{A} = \mathcal{F}^G\,, \qquad \mathrm{DHR}(\mathcal{A}) \overset{\mathrm{br}}{\cong} \mathrm{Rep}^k(G)$$

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Subnets of observables:

- If $\mathcal{B} \subset \mathcal{A}$ an irreducible subnet, then $\exists!$ $H \geq G$ and $\mathcal{B} = \mathcal{F}^H$.
- If $G = N \leq H$ normal, then $\mathcal{B} = \mathcal{A}^{H/N}$

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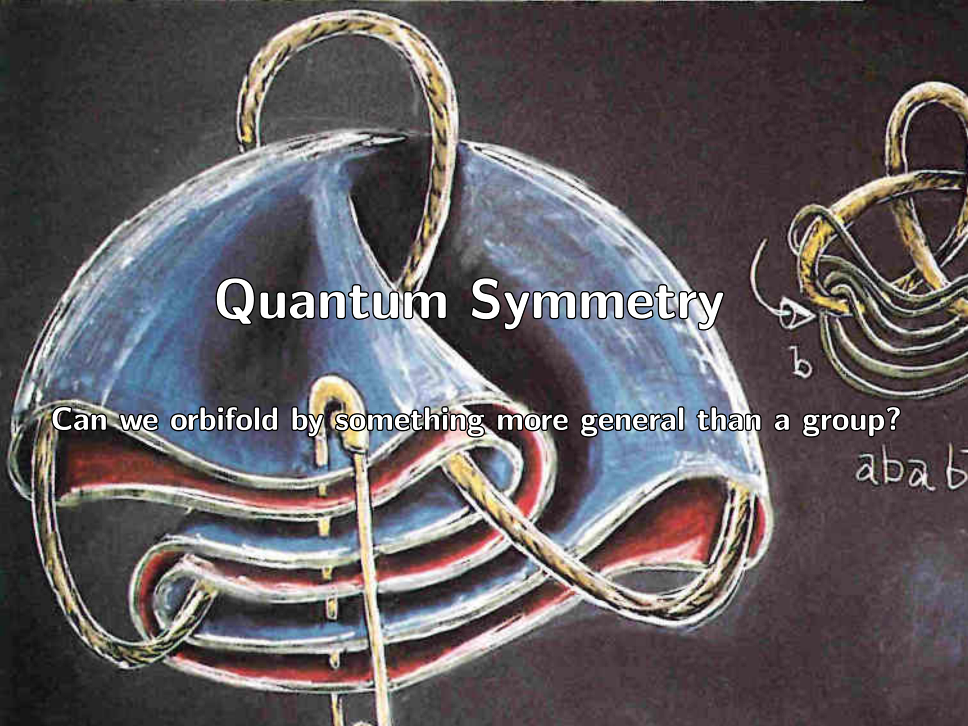
Slogan

Global symmetries and superselection rules are completely described by (super-)groups.

Most of this fails for low-dimensional QFT.



Quantum Symmetry

Can we orbifold by something more general than a group?

A **local (conformal) net** on $S^1 \cong \mathbb{R} \cup \{\infty\}$ is a map

$$\mathbb{R} \supset I \longmapsto \mathcal{A}(I) \subset \mathrm{B}(\mathcal{H})$$

fulfilling a bunch of axioms including:

- **Isotony:** $\mathcal{A}(I) \subset \mathcal{A}(J)$ for $I \subset J$
- **Haag duality:** $\mathcal{A}(I') = \mathcal{A}(I)'$, where $I' = \mathbb{R} \setminus \overline{I} \rightsquigarrow$ **locality:** $[\mathcal{A}(I), \mathcal{A}(J)] = \{0\}$ for $I \subset J'$.
- **Covariance:** $U\colon G \to \mathcal{U}(\mathcal{H})$, s.t. $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)$.
- **Vacuum:** Unique (up to phase) G-invariant unit vector $\Omega \in \mathcal{H}$, s.t. $\overline{\vee_I \mathcal{A}(I)\Omega} = \mathcal{H}$.

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Motivation

- axiomatizes Unitary Chiral Conformal Field Theory
- describes edge of Topological Phases of Matter, Topological Quantum Computing
- 3-manifold invariants, 3-2-1 Topological Field Theories, 2+1d Quantum Gravity (Witten)

A **quantum operation** $\phi \in \mathrm{QuOp}(\mathcal{A})$ on $\mathcal{A}$ is a family $\phi = \{\phi_I \colon \mathcal{A}(I) \to \mathcal{A}(I)\}$ with

- **Compatible:** $\phi_J \restriction \mathcal{A}(I) = \phi_I$ for $I \subset J$
- **Unital completely positive**, i.e. $\phi_I(1) = 1$ and $\phi_I \otimes \mathrm{id} \colon \mathcal{A}(I) \otimes M_n(\mathbb{C}) \to \mathcal{A}(I) \otimes M_n(\mathbb{C})$ is positive for all $n \in \mathbb{N}$.
- **Vacuum preserving** $(\Omega, a\Omega) = (\Omega, \phi_I(a)\Omega)$ for $a \in \mathcal{A}(I)$.
- **Extremal**, i.e. $\phi_I = \lambda\psi_1 + (1-\lambda)\psi_2$ for some $\lambda \in (0,1)$ and ψ_1, ψ_2 vpucp then $\psi_1 = \psi_2 = \phi_I$.
- Ω-**Markov:** there is an Ω-adjoint $\phi_I^\sharp$ with $(\phi_I^\sharp(a)\Omega, b\Omega) = (a\Omega, \phi_I(b)\Omega)$ for $a, b \in \mathcal{A}(I)$.

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Generalization of the group of global gauge automorphisms:

$$\mathrm{Aut}(\mathcal{A}) = \mathrm{QuOp}(\mathcal{A})^\times \subset \mathrm{QuOp}(\mathcal{A})$$

Notation:

- $Q = \{\phi_0, \dots, \phi_n\}$ finite set, $\mathbb{C}Q$ free vector space over K
- $\mathrm{Conv}(Q) = \{\sum_{i=0}^n \lambda_i \phi_i \in \mathbb{C}Q : \lambda_i \in [0,1] \text{ and } \sum_{i=0}^n \lambda_i = 1\}$
- $\phi_i \prec \sum_{k=0}^n \lambda_k \phi_k \in \mathrm{Conv}(Q)$ if and only if $\lambda_i > 0$.

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Definition

A **(finite) hypergroup** is a set $Q = \{\phi_0, \ldots, \phi_n\}$ with an evolution $i \mapsto \bar{i}$ and a structure of an associative unital $*$-algebra structure on $\mathbb{C}Q$:

$$\phi_i \circ \phi_j = \sum_{k=0}^{n} C_{ij}^k \phi_k \,, \qquad \phi_i^* = \phi_{\bar{i}} \,, \qquad \text{with identity } \phi_0 = 1\text{, such that}$$

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Then $w_k = (C_{k\bar{k}}^0)^{-1} \geq 1$ is called **weight** and $D(Q) = \sum_k w_k$ **global weight**.

Examples of hypergroups:

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$$c_{\pi_1} \circ c_{\pi_2} = \sum_{\pi \in \mathrm{Irrep}(\hat{G})} \dim \mathrm{Hom}_G(\pi_1 \otimes \pi_2, \pi) \frac{d\pi}{d\pi_1 d\pi_2} \cdot c_\pi$$

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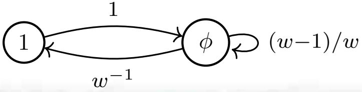
- One parameter deformation $\mathbb{Z}_2^q = \{1, \phi\}$ of $\mathbb{Z}_2$ with $q \in [0, 1)$

$$\phi \circ \phi = (1 - q) \cdot 1 + q \cdot \phi \qquad w = q^{-1}$$



Theorem ((B. '16))

Let $K \subset \mathrm{QuOp}(\mathcal{A})$, *then* $\mathcal{A}^K(I) = \{a \in \mathcal{A}(I) : \phi_I(a) = a$ *for all* $\phi \in K\}$ *defines a subnet* $\mathcal{A}^K \subset \mathcal{A}$.

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Theorem (Galois correspondence for Conformal Nets (B. '16))

Let $\mathcal{A}$ *be a local conformal net. There is a one-to-one correspondence*

$$\boxed{\begin{array}{c}\mathcal{B} \subset \mathcal{A} \\ \textit{finite index}\end{array}} \longleftrightarrow \boxed{\begin{array}{c}Q \subset \mathrm{QuOp}(\mathcal{A}) \\ \textit{hypergroup}\end{array}}$$

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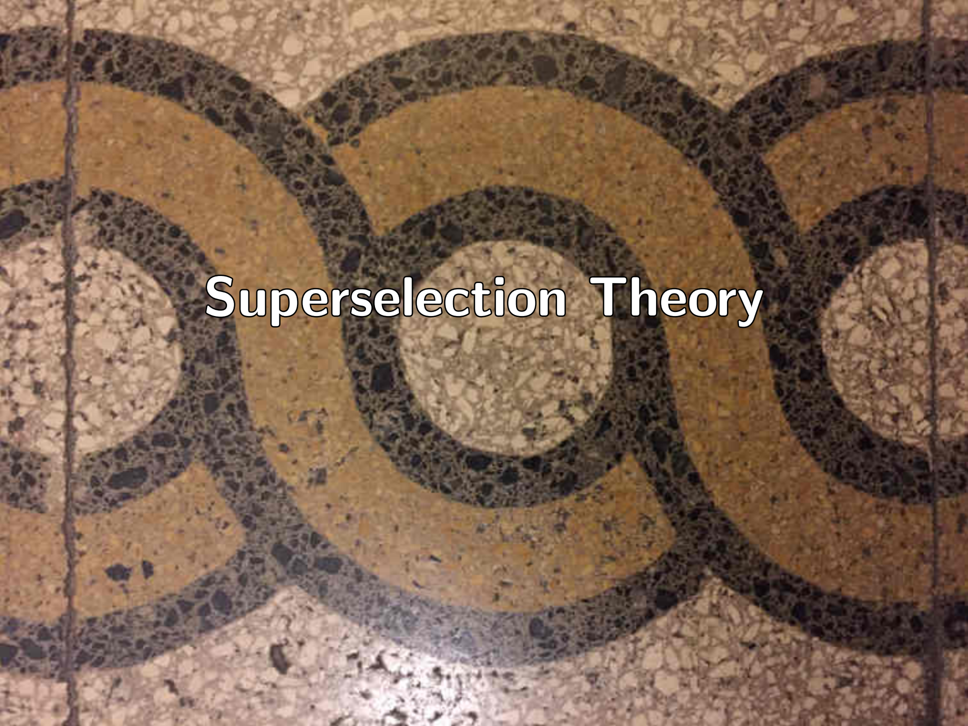
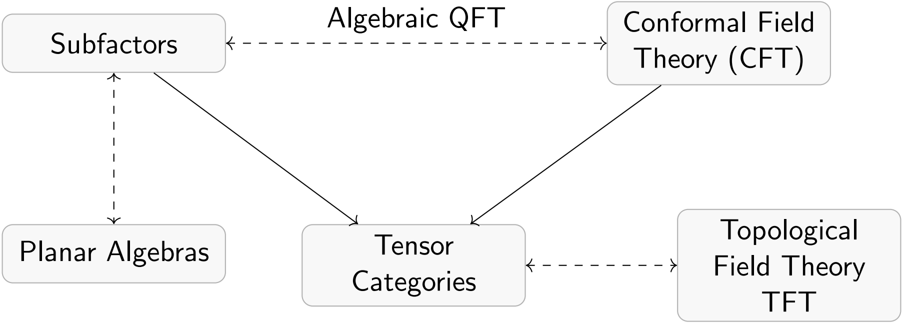
Corollary ((B. '16))

If H *finite Hopf algebra acting on* $\mathcal{A}$*, then* $H = \mathbb{C}G$ *for a finite group* G*.*



Superselection Theory



In binding together elements long-known but heretofore scattered and appearing unrelated to one another, it suddenly brings order where there reigned apparent chaos — Henri Poincaré

A **representation** π of a net $I \mapsto \mathcal{A}(I)$ is a family of representations

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Example

The **vacuum representation:** $\mathrm{id} = \{\mathrm{id}_{\mathcal{A}(I)} \colon \mathcal{A}(I) \to \mathcal{B}(\mathcal{H})\}$.

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- Unitary equivalence class $[\pi]$ is a superselection sector $=$ charge
- localized endomorphisms can be composed $\leftrightarrow$ $\otimes$-product structure $\rightsquigarrow$ "addition" of irreducible charges:

$$\rho \circ \sigma \cong \bigoplus_\tau N^\tau_{\rho,\sigma} \tau\,, \qquad N^\tau_{\rho,\sigma} \in \{0,1,2,\ldots\}$$

where $N^\tau_{\rho,\sigma} \in \{0,1,2,\ldots\}$ and $[\rho],[\sigma],[\tau]$ are irreducible charges.

Braiding due to (Fredenhagen–Rehren–Schroer '89)

It holds $\rho^I \circ \sigma^J = \sigma^J \circ \rho^I$ for $I < J$ or $(J < I)$.

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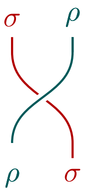
$$\varepsilon_{\rho,\sigma}\, \rho \circ \sigma(\,\cdot\,) = \rho \circ \sigma(\,\cdot\,)\, \varepsilon_{\rho,\sigma}$$

fixed by asking $\varepsilon_{\rho^I,\sigma^J} = 1$ for $I > J$.



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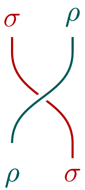

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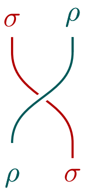

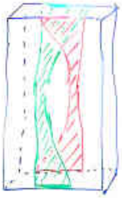

$\rightsquigarrow$ **Yang–Baxter** relation, **braid group** representations, . . .

- A net $\mathcal{A}$ is called **rational** if it has only finitely many irreducible equivalence classes (sectors) $\mathrm{Irr}(\mathrm{Rep}(\mathcal{A}))$ with finite quantum dimension, i.e. $\mathrm{Rep}(\mathcal{A})$ is a unitary fusion category.

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Theorem ((Kawahigashi–Longo–Müger '01), (B. '16))

$\mathcal{A}$ rational and $Q \subset \mathrm{QuOp}(\mathcal{A})$ finite hypergroup, then $\mathcal{A}^Q$ is rational and $\mathrm{Dim}(\mathrm{Rep}(\mathcal{A}^Q)) = D(Q)^2\,\mathrm{Dim}(\mathrm{Rep}(\mathcal{A}))$.

Twisted representations (after Müger)

- Let $\alpha \in \mathrm{Aut}(\mathcal{A})$. An α-twisted representation is a representation which is α-localized in some interval I, i.e. for every $I_1 < I < I_2$ we have

$$\rho_{I_1} = \mathrm{id}_{\mathcal{A}(I_1)}\,, \qquad \rho_{I_2} = \alpha_{I_2}$$

- Let $G \leq \mathrm{Aut}(\mathcal{A})$, then we have the category $G\text{–}\mathrm{Rep}(\mathcal{A})$ generated by α-twisted representations with $\alpha \in G$.
- (Müger '05) This category has an action of G and we can form the equivariantization $G\text{–}\mathrm{Rep}(\mathcal{A})^G$ which is braided equivalent to $\mathrm{Rep}(\mathcal{A}^G)$.
- (Dijkgraaf–Pasquier–Roche '90), (Müger '10)
  Let $G \leq \mathrm{Aut}(\mathcal{A})$ a finite group, $\mathcal{A}$ holomorphic then $\mathrm{Rep}(\mathcal{A}^G) \cong \mathrm{Rep}(D^\omega(G))$ (twisted quantum double)

Similarly, we can define $Q\text{–}\mathrm{Rep}(\mathcal{A})$ for every hypergroup $Q \subset \mathrm{QuOp}(\mathcal{A})$.

Theorem ((B. '16))

If $\mathcal{A}$ is holomorphic, then $Q\text{–}\mathrm{Rep}(\mathcal{A})$ is a **categorification** *of Q, i.e. $K_{Q\text{–}\mathrm{Rep}(\mathcal{A})} \cong Q$ and superselection theory is given by* **Drinfel'd center** *aka* **quantum double***:*

$$\mathrm{Rep}(\mathcal{A}^Q) \overset{\mathrm{br}}{\cong} Z(Q\text{–}\mathrm{Rep}(\mathcal{A}))\,.$$

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Interpretation: The category of Q-equivariant Q-twisted representations:

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Question

Do all unitary fusion categories arises as $Q\text{–}\mathrm{Rep}(\mathcal{A})$ for some conformal net $\mathcal{A}$?

If true $\rightsquigarrow$ all unitary 3-2-1-0 extended TFTs come from Conformal Nets.

Theorem ((B. '16))

Let $\mathcal{A}$ be rational and $Q \subset \mathrm{QuOp}(\mathcal{A})$ hypergroup.

- $Q \cong K_{Q\text{–}\mathrm{Rep}(\mathcal{A})} /\!\!/ K_{\mathrm{Rep}(\mathcal{A})}$ *and*

$$\begin{tikzcd} & Z(Q\text{–}\mathrm{Rep}(\mathcal{A})) \arrow[d, two heads, "\mathrm{forget}"] \\ \overline{\mathrm{Rep}(\mathcal{A})} \arrow[ur, hook, dashed, "\mathrm{br}"] \arrow[r, hook, "\iota"] & Q\text{–}\mathrm{Rep}(\mathcal{A}) \end{tikzcd}$$

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Drinfel'd center formula ($\otimes$-categorical abstract non-sense) (Ocneanu, Böckenhauer–Evans–Kawahigashi, Davydov–Müger–Nikshych–Ostrik):

$$\rightsquigarrow Z(Q-\mathrm{Rep}(\mathcal{A})) \overset{\text{br}}{\cong} \mathrm{Rep}(\mathcal{A}^Q) \boxtimes \overline{\mathrm{Rep}(\mathcal{A})}\,,$$

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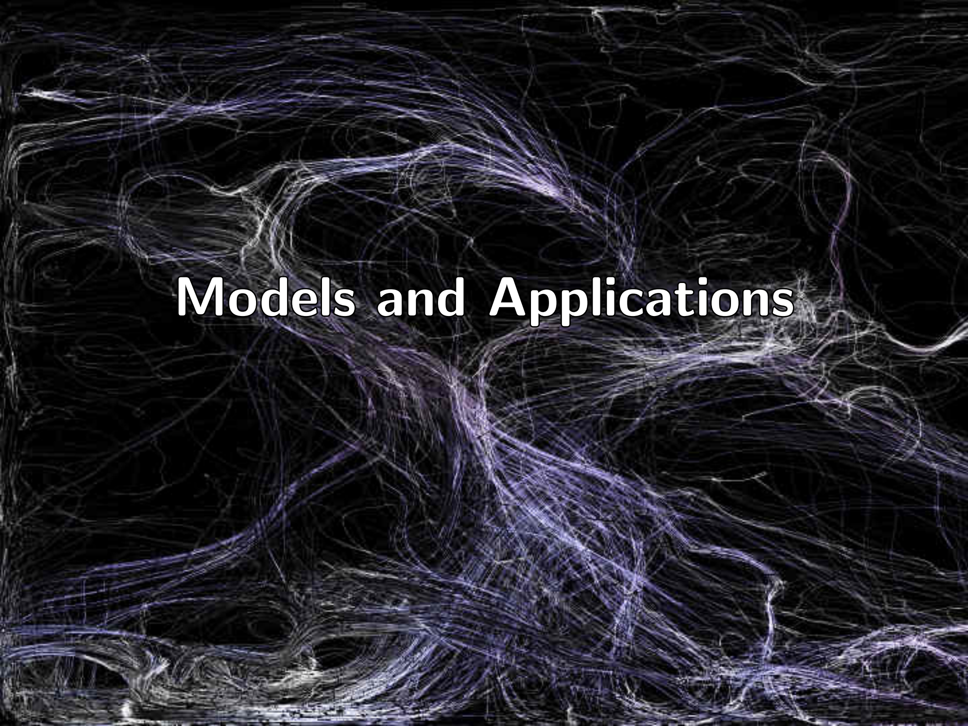
Interpretation: $Q\text{–}\mathrm{Rep}(\mathcal{A})$ is a Q-hypergraded extension of $\mathrm{Rep}(\mathcal{A})$ and

$$\mathrm{Rep}(\mathcal{A}^Q) \overset{\mathrm{br}}{\cong} (Q\text{–}\mathrm{Rep}(\mathcal{A}))^Q \qquad := \overline{\mathrm{Rep}(\mathcal{A})}' \cap Z(Q\text{–}\mathrm{Rep}(\mathcal{A}))\,.$$

(Müger centralizer)

Models and Applications



Chiral Wess–Zumino–Witten model (χWZW) given by:
Loop group net of G at level k. G compact Lie group, $\mathrm{L}G = C^\infty(S^1, G)$

$$\mathcal{A}_{G_k}(I) = \pi_{0,k}(\mathrm{L}_I G)'', \qquad \mathrm{L}_I G = \{\gamma \in \mathrm{L}G : \operatorname{supp}\gamma \subset I\}$$

gives a net on S^1 and by restriction a net on $\mathbb{R} \cong S^1 \setminus \{-1\}$.

Example ($\mathrm{Rep}(\mathcal{A}_{\mathrm{SU}(2)_k})$ (Wassermann))

Irreducible representations $\{0, \frac{1}{2}, 1, \ldots, \frac{k}{2}\}$:

$$[i] \times [j] = \bigoplus_{n=|i-j|}^{\min(i+j, k-i-j)} [n]$$

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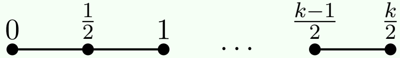
$\mathrm{Rep}(\mathcal{A}_{\mathrm{SU}(2)_k})$ is $\otimes$-generated by $\frac{1}{2}$-representation $\rho_{\frac{1}{2}}$.

A_{k+1} :



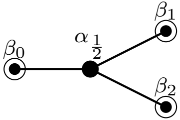
Statistical dimension:

$$d\rho_{\frac{1}{2}} = 2\cos\left(\frac{\pi}{k+2}\right)$$

Braiding: given essentially by the Jones polynomial at some root of unity.

Consider $\mathcal{A} = \mathcal{A}_{\mathrm{SU}(2)_4} \subset \mathcal{B} = \mathcal{A}_{\mathrm{SU}(2)_4} \rtimes \mathbb{Z}_2 = \mathcal{A}_{\mathrm{SU}(3)_1}$

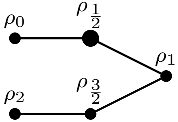
$D_4 = 1^{\mathbb{Z}_3}$:



$\mathcal{A}_{\mathrm{SU}(3)_1}$

$\cup$

A_5 :



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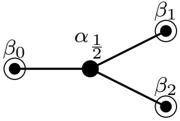

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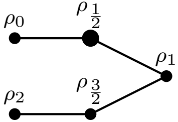
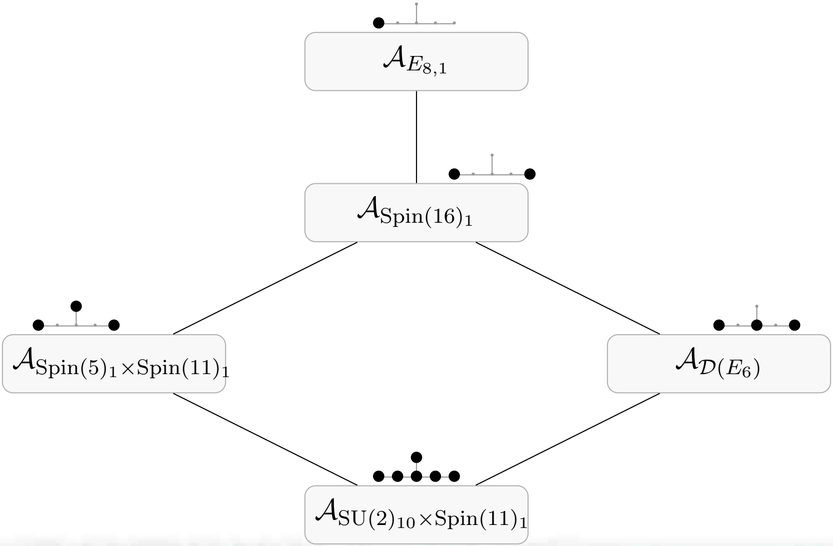


$\mathcal{A}_{\mathrm{SU}(2)_4}$

- β_i are sectors of $\mathrm{Rep}(\mathcal{A}_{\mathrm{SU}(3)_1})$ with $\mathbb{Z}_3$–fusion rules.
- $\alpha_{\frac{1}{2}}$ is a soliton with dimension $\sqrt{3}$.

Theorem ((B.'16+))

All 1^A tensor categories with A abelian group and $|A|$ odd arise as $\mathbb{Z}_2$–$\mathrm{Rep}(\mathcal{A})$ for some lattice (= torus loop group) conformal net $\mathcal{A}_{\mathbb{T}^n{}_L}$.

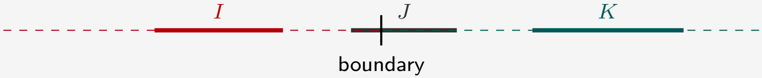

$\mathcal{A}_{E_{8,1}}$
$\mathcal{A}_{\mathrm{Spin}(16)_1}$
$\mathcal{A}_{\mathrm{Spin}(5)_1 \times \mathrm{Spin}(11)_1}$
$\mathcal{A}_{\mathcal{D}(E_6)}$
$\mathcal{A}_{\mathrm{SU}(2)_{10} \times \mathrm{Spin}(11)_1}$

Definition

Let $\mathcal{B}_{\mathrm{L}}, \mathcal{B}_{\mathrm{R}} \supset \mathcal{A}$. An $\mathcal{A}$-**topological** $\mathcal{B}_{\mathrm{L}}$–$\mathcal{B}_{\mathrm{R}}$ **defect** (or phase boundary) are extensions $\mathcal{A} \subset \mathcal{B}_{\mathrm{L}}, \mathcal{B}_{\mathrm{R}} \subset \mathcal{D}$ on the same Hilbert space:

$$
\begin{array}{rcr}
[\mathcal{B}_{\mathrm{L}}(I), \mathcal{D}(J)] = \{0\} & I < J & \mathcal{B}_{\mathrm{L}} \text{ is left local wrt } \mathcal{D} \\
[\mathcal{B}_{\mathrm{R}}(K), \mathcal{D}(J)] = \{0\} & K > J & \mathcal{B}_{\mathrm{R}} \text{ is right local wrt } \mathcal{D} \\
\Longrightarrow [\mathcal{B}_{\mathrm{L}}(I), \mathcal{B}_{\mathrm{R}}(K)] = \{0\} & I < K & \mathcal{B}_{\mathrm{R}} \text{ is right local wrt } \mathcal{B}_{\mathrm{L}}
\end{array}
$$

and $\mathcal{D}(J) = \mathcal{B}_{\mathrm{L}}(J) \vee \mathcal{B}_{\mathrm{R}}(J)$



Describes a topological defect (invisible for $\mathcal{A}$) between $\mathcal{B}_{\mathrm{L}}$ and $\mathcal{B}_{\mathrm{R}}$.

Let $\mathcal{A}$ be holomorphic.

- We say $G \leq \mathrm{Aut}(\mathcal{A})$ if is **anomaly free** if the associated $[\omega] \in H^3(G, \mathbb{T})$ is trivial.

$\rightsquigarrow$ We can form $\mathcal{A} \rtimes G$ (choice of $H^2(G, \mathbb{T})$) which is an $\mathcal{A}^G$–topological defect between $\mathcal{A}$ and $\mathcal{A}/\!\!/G$.

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Example

Let Γ be the Leech lattice, then there is a holomorphic conformal net $\mathcal{A}_\Gamma = \mathcal{A}_{\mathbb{R}^{24}/\Gamma}$ and the reflection gives anomaly free $\mathbb{Z}_2 \leq \mathrm{Aut}(\mathcal{A}_\Gamma)$. Moonshine net (Kawahigashi–Longo '06)[a]

$$\mathcal{A}^\natural := \mathcal{A}_\Gamma /\!\!/ \mathbb{Z}_2$$

$$\mathrm{Aut}(\mathcal{A}^\sharp) \cong \mathbb{M} \text{ the } \textbf{Monster group} \text{ with } |\mathbb{M}| \approx 8 \cdot 10^{53}$$

[a] assoc. w. the Frenkel–Lepowski–Meurman Moonshine Vertex Algebra $V^\sharp$

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Mathematical Physics $\rightsquigarrow$ Pure Mathematics

$\langle[\omega]\rangle \cong \mathbb{Z}_{24} \leq H^3(\mathbb{M}, \mathbb{T})$ (Johnson-Frey '17) $\qquad \langle[\omega]\rangle \stackrel{?}{=} H^3(\mathbb{M}, \mathbb{T})$

If G is non-abelian then the "dual" $\hat{G}$ (more precisly $(\mathbb{C}G)^*$) is only a Hopf/Kac algebra. $\exists$ **reverse twisted orbifold** $\mathcal{B}/\!\!/\hat{G}$ with

$$\mathcal{A} \xrightarrow{(\,\cdot\,)/\!\!/G} \mathcal{A}/\!\!/G =: \mathcal{B} \xrightarrow{(\,\cdot\,)/\!\!/\hat{G}} \mathcal{B}/\!\!/\hat{G} = \mathcal{A} \quad ?$$

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We say a finite-dim Hopf/Kac algebra H acts **anomaly free** on $\mathcal{A}$ if $\exists$ $Q \leq \mathrm{QuOp}(\mathcal{A})$ with $Q \cong K_{\mathrm{CoRep}(H)}$ and $Q\text{--}\mathrm{Rep}(\mathcal{A}) \overset{\otimes}{\cong} \mathrm{Rep}(H)$.

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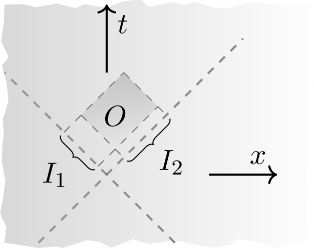
Theorem ((B. (unpublished)))

If H acts anomaly free on $\mathcal{A}$ $\exists$ holomorphic net $\mathcal{A}/\!\!/H$, such that $\mathcal{A} \rtimes H$ is a $\mathcal{A}^Q$ topological defect between $\mathcal{A}$ and $\mathcal{A}/\!\!/H$, namely

$$\mathcal{A}/\!\!/H(a,b) = (\mathcal{A} \rtimes H)(a,b) \cap (\mathcal{A} \rtimes H)(-\infty, a)'$$

Further, $\hat{H}$ acts anomaly free on $\mathcal{A}/\!\!/H$ and $\mathcal{A}/\!\!/H/\!\!/\hat{H} = \mathcal{A}$.



One can define a **conformal net** on **Minkowski space** by

$$\mathcal{A}_2(O) = \mathcal{A}_+(I_1) \otimes \mathcal{A}_-(I_2)$$

where $\mathcal{A}_\pm$ are conformal nets on $\mathbb{R}$.

Full CFTs based on $\mathcal{A}_\pm$ completely rational are given by maximal local extensions

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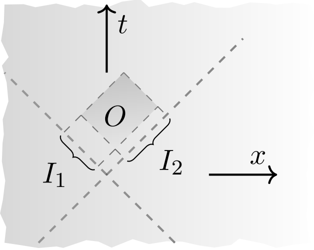
such that $\mathcal{B}_2$ has only the vacuum sector.

- **Locality**. $[\mathcal{B}_2(O_1), \mathcal{B}_2(O_2)] = \{0\}$ if O_1 and O_2 are space like separated.



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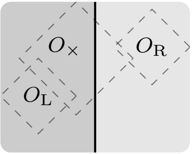
- **Locality**. $[\mathcal{B}_2(O_1), \mathcal{B}_2(O_2)] = \{0\}$ if O_1 and O_2 are space like separated.
- Physically, the conformal net $\mathcal{A}_2$ describes (generalized) **symmetries** of the full CFT $\mathcal{B}_2$.



$$\begin{Bmatrix} O_{\mathrm{L}} \\ O_\times \\ O_{\mathrm{R}} \end{Bmatrix} \longmapsto \begin{Bmatrix} \mathcal{B}_{\mathrm{L}}(O_{\mathrm{L}}) \\ \mathcal{D}(O_\times) \\ \mathcal{B}_{\mathrm{R}}(O_{\mathrm{R}}) \end{Bmatrix} \supset (\mathcal{A}_2)(O_\bullet)$$

- Defect line invisible for the subnet $\mathcal{A}_2$ (conserves symmetries prescribed by $\mathcal{A}$)
- Different realization $\leftrightarrow$ different boundary conditions
- $\mathcal{A}$-topological $\mathcal{B}_{\mathrm{L}}$-$\mathcal{B}_{\mathrm{R}}$ defect line.

Theorem ((B–Kawahigashi–Longo–Rehren '15),(B–Kawahigashi–Longo–Rehren '16))

Let $\mathcal{B}_{\mathrm{L}}, \mathcal{B}_{\mathrm{R}}$ maximal local extensions of $\mathcal{A} \otimes \mathcal{A}$ and $\mathcal{A}$ completely rational. Then $\mathcal{A} \otimes \mathcal{A}$–topological $\mathcal{B}_{\mathrm{L}}$–$\mathcal{B}_{\mathrm{R}}$ phase boundaries can be classified from the categorical data associated with $\mathrm{Rep}(\mathcal{A})$ as in (Fröhlich–Fuchs–Runkel–Schweigert '07)

Theorem ((B–Kawahigashi–Longo–Rehren '15),(B–Kawahigashi–Longo–Rehren '16))

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Theorem ((B.))

Assume $Q \subset \mathrm{QuOp}(\mathcal{A})$ is a finite hypergroup and $\mathcal{A}$ rational conformal net on S^1.
Consider $\mathcal{A}^Q \otimes \mathcal{A} \subset \mathcal{A} \otimes \mathcal{A} \subset \mathcal{B}_2$, where $\mathcal{A} \otimes \mathcal{A} \subset \mathcal{B}_2$ is the canonical Longo–Rehren extension (Cardy case).
Then there is a one-to-one correspondence between $\mathcal{A}^Q \otimes \mathcal{A}$-topological $\mathcal{B}_2$-$\mathcal{B}_2$ defects and sectors in Q–$\mathrm{Rep}(\mathcal{A})$.

Kramers–Wannier duality of the conformal Ising model:
The unique full CFT $\mathcal{B}_2 \supset \mathrm{Vir}_{1/2} \otimes \overline{\mathrm{Vir}}_{1/2}$ has a duality defect which gives rise to the duality

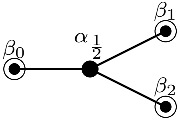
$$(1, \sigma, \varepsilon) \longleftrightarrow (1, \mu, -\varepsilon)$$

Generalizes to a duality defect $\alpha_{\frac{1}{2}}$ for $\mathcal{B}_2 \supset \mathcal{A}^{\mathbb{Z}_2}_{\mathrm{SU}(3)_1} \otimes \bar{\mathcal{A}}_{\mathrm{SU}(3)_1}$:

$$(1, \sigma_{\chi_1}, \sigma_{\chi_2}, \varepsilon) \longleftrightarrow (1, \mu_{\chi_1}, \mu_{\chi_2}, -\varepsilon)$$

with $\hat{\mathbb{Z}}_3 = \{1, \chi_1, \chi_2\}$. Remember $\mathcal{A}^{\mathbb{Z}_2}_{\mathrm{SU}(3)_1} = \mathcal{A}_{\mathrm{SU}(2)_4} \subset \mathcal{A}_{\mathrm{SU}(3)_1}$

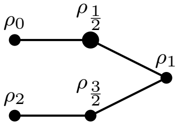
$D_4 = 1^{\mathbb{Z}_3}$:



$\mathcal{A}_{\mathrm{SU}(3)_1}$

$\cup$

A_5 :



$\mathcal{A}_{\mathrm{SU}(2)_4}$

More general, for every 1^A (Tambara–Yamagami) fusion category.

Generalized Orbifolds

- Find all finite hypergroups in $\mathrm{QuOp}(\mathcal{A}_{E_{8,1}})$, construct the hypothetical Haagerup CFT of (Evans–Gannon '11).
- Infinite, e.g. compact hypergroup actions $\rightsquigarrow$ analytical and approximation properties
- Is $\mathrm{QuOp}(\mathcal{A})$ always a compact hypergroup and $\mathcal{A}^{\mathrm{QuOp}(\mathcal{A})} = \mathrm{Vir}_{\mathcal{A}}$?
- What is $\mathrm{Rep}(\mathrm{Vir}_c)$ for $c \geq 1$ and can we classify all rational conformal nets by a nice structure similar to $\mathrm{Rep}(\mathrm{TLJ})$?

Reconstruction Program:

- Given algebraic data of **planar algebra**, **fusion category** or **subfactor** construct a conformal net such that defects revover the **planar algebra**, **fusion category** or **subfactor**, respectively.

Thank you for your attention!

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