Positive, self-adjoint and additive Wick's polynomials of scalar fields

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(Work in progress in collaboration with K. Fredenhagen and S. Hollands)

"Quisquis huc accedis: quod tibi horridum videtur mihi amoenum est; si placet, maneas, si taedet abeas, utrumque gratum" Villa Farnesina, Accademia dei Lincei

Outline

Introduction

Quantum field theory

- In quantum physics the physical degrees of freedom are represented in a non commutative algebra of operators on a Hilbert space.
- In a naive Ansatz, quantum fields are thought of as operator valued functions, but due to unavoidable singularities a better concept is that of operator-valued distributions (Gårding and Wightmann, 1964).
- This allows a precise formulation of a quantum field theory with linear equations (free fields).

- The commutation relations of a free scalar field

$$[\varphi(x), \varphi(y)] = i\Delta(x - y) \ ,$$

can be understood in terms of operator-valued distributions as a relation in a unital algebra ($f, g \in \mathcal{D}(M)$)

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Fundamental Question
How can nonlinear relations between fields be formulated?

Wick's ordering for Wick's polynomials

A formal solution to the problem is known on Minkowski spacetime since long (Gian-Carlo Wick, 1950) by Wick's ordering, namely we decompose the field in terms of creation and annihilation operators on Fock space

$$\varphi(x) = a^\dagger(x) + a(x)$$

and define the Wick's ordered powers of the field as follows

$$:\varphi^n(x):= \sum_{k=0}^{n} \binom{n}{k} a^\dagger(x)^k a(x)^{n-k}$$

Gårding and Wightmann (1964) proved that these are well defined operator-valued distributions with a dense domain $\mathcal{D}_{GW}$ on Fock space.

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Invoking PCT transformation one learns at least that they all have equal defect indices (0 or ∞) (Driessler, Summers, Wichmann, 1986, or Buchholz, Fredenhagen, private communication), so they have selfadjoint extensions, or by enlarging the Hilbert space one gets extensions as well (Borchers, Yngvason, 1991).

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But, besides selfadjointness several other crucial questions remained open...

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OQ3. What is the relation with the Weyl C^*-algebra (or the Buchholz-Grundling resolvent algebra) generated by the field $W(f) = e^{i\varphi(f)}$?

Math for the Wick's polynomials on a natural invariant domain

Natural and invariant domain $\mathcal{D}_\mu$

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Consider the following nonlinear map, ($\mathsf{W}(\mathsf{f}) \in \mathcal{W}(\mathsf{M})$)

$$\mathcal{D}(\mathsf{M}) \ni \mathsf{f} \mapsto \mathsf{G}_\phi(\mathsf{f}) \doteq \mathsf{e}^{\frac{1}{2}\omega_2(\mathsf{f},\mathsf{f})}\pi_\omega(\mathsf{W}(\mathsf{f}))\phi\,, \qquad \phi \in \mathcal{H}_\omega\,.$$

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Consider the following nonlinear map, $(W(f) \in \mathcal{W}(M))$

$$\mathcal{D}(M) \ni f \mapsto G_\phi(f) \doteq e^{\frac{1}{2}\omega_2(f,f)}\pi_\omega(W(f))\phi\ , \qquad \phi \in \mathcal{H}_\omega\ .$$

Definition (Smooth maps)
We define G_ϕ to be infinitely differentiable (at 0) whenever it is continuous and the following maps (functional derivatives)

$$\frac{\delta^k G_\phi}{\delta f^k}[0] : \underbrace{\mathcal{D}(M) \times \cdots \times \mathcal{D}(M)}_{\text{k-times}} \mapsto \mathbf{C}$$

exists and are symmetric, $k-$linear and continuous in the respective topologies for any natural $k \geq 1$ (hence they are distributions!)

Natural and invariant domain $\mathcal{D}_\mu$

For our case we define, in the sense of distributions (valued in $\mathcal{H}_\omega$),

$$\frac{\delta^k G_\phi}{\delta f^k}[0](x_1, \ldots, x_k) \doteq \iota^k : \varphi(x_1) \cdots \varphi(x_k) : \phi \, .$$

Definition (Microlocal domain of smoothness)

$\phi \in \mathcal{H}_\omega$ is in $\mathcal{D}_\mu$ if $f \mapsto G_\phi(f)$ is infinitely differentiable (at 0 for instance) and the distributions $\frac{\delta^k G_\phi}{\delta f^k}[0]$ have wave front sets as (for any $k \geq 1$)

$$\mathrm{WF}\left(\frac{\delta^k G_\phi}{\delta f^k}[0]\right) \subseteq \{(x_1, p_1; \cdots ; x_k, p_k) \in \dot{T}^* M^k \mid p_j \in \overline{V}_+ \, , j = 1, \ldots, k\}$$

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> "$C^\infty(H)$ is a core for $(:\varphi^k:, \mathcal{D}_\mu)$"

Notice: Crucial that, by the third result, $\mathcal{D}_\mu$ depends only on the representation and not the state ω (cyclic vector).

Positive Wick's polynomials

Positivity

We know that a positive power of (any) Wick's monomial does not mean that the operator is positive (in Hilbert space), for instance $:\varphi^2(f):$ is never a positive operator even if f is a nonnegative test function! (Epstein, Glaser, Jaffe, 1965) The reason is simple, we know that

$$\langle \Omega_\omega, :\varphi^2(f): \Omega_\omega \rangle = 0$$

but $:\varphi^2(f): \Omega_\omega \neq 0$. Hence, the following function

$$\mathbf{R} \ni \lambda \mapsto F(\lambda) = \langle (1 + \lambda :\varphi^2(f):)\Omega_\omega, :\varphi^2(f): (1 + \lambda :\varphi^2(f):)\Omega_\omega \rangle$$

has a zero $F(0) = 0$ but $F'(0) = 2\| :\varphi^2(f): \Omega_\omega\|^2 > 0$, hence $F(\lambda)$ gets negative values!

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One possible consequence is that the energy density is no longer positive, and the positive energy condition of general relativity does not hold anymore, hence we may get exotic spacetimes (wormholes, warp drives, etc...)

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But, let me generalize this a bit!

Using Borchers (1960) we know that it is sufficient to restrict the attention to the time axis, hence we define with $f \in \mathcal{D}(\mathbf{R}, \mathbf{R})$,

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(otherwise, use microlocal arguments...).

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The crucial trick is that the two-point massless function dominates, as a positive distribution, the massive two-point function, i.e.

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plus the symmetry of the Wick's product, Wick's Theorem, some distributional identities, microlocal arguments, and the explicit form of the two-point functions, at the end one obtains ($c_0, c_1 > 0$)

$$A_4(f) \doteq :\varphi^4(f^2): + c_0 \int dt :(\partial_t(\varphi(t)f(t)))^2: + c_1 \int dt\, \dot{f}(t)^2 \geq 0$$

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With more work and pain, it is also true that for any even power that ($c_k > 0$)

$$A_{2n}(f) \doteq \sum_{k=0}^{n} c_k \int dt \; :\Big(\partial_t^k[\varphi^{n-k}(t)f(t)]\Big)^2: \geq 0 \,,$$

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is a positive operator on $\mathcal{D}_\mu$! However, they are not Wightmann fields...

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such that they satisfy the following additive relation

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Additivity is equivalent to being simply localized, i.e., can be written as a finite sum of additive maps but localized in arbitrarily small regions (as it happens for the linear case by partition of unity).

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- Even in the case of the Friedrichs extension the situation does not seem to be clear...

Something more refined is necessary!

Quadratic form construction

First steps

Consider the massless situation first and define

$$a_f(\phi,\phi) = \frac{1}{\pi}\int_0^\infty d\alpha \left\| \frac{:\varphi^n(f_\alpha):}{n!}\phi \right\|^2$$

where $f_\alpha(t) = f(t)e^{i\alpha t}$.

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where $f_\alpha(t) = f(t)e^{i\alpha t}$.
By some straightforward manipulations (similar to those seen before) we get that

$$a_f(\phi, \phi) = \langle \phi, A_{2n}(f)\phi \rangle \ !!!$$

So, we get back to the operators we constructed earlier but for the massless field. But since the quadratic form is symmetric, positive by definition, closable (since defined via a symmetric linear operator), if we consider its closure, then it has a unique selfadjoint and positive extension, hence there exists a unique selfadjoint and positive operator inducing that form. Since the operator A_{2n} is additive, is the extension (via the form) $\overline{A}_{2n}$ also additive?

Form additivity

Clearly, additivity should hold in the sense of forms, namely, for any extended non linear map $f \mapsto \overline{A}(f)$ into positive and selfadjoint operators (Wick's polynomials), one wishes that

$$\overline{A}(f+g+h) \dotplus \overline{A}(g) = \overline{A}(f+g) \dotplus \overline{A}(g+h)\ .$$

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The problem is that if we strictly repeat the same calculations done previously we end up with nonlocal polynomials due to the form of the two-point function in the massive case, for instance, we saw the bound of Fewster for the square of φ and its form is the following, written in terms of the integration over α

$$C(f) = \mathrm{const.} \int \frac{d^3\vec{k}}{2\omega(\vec{k})} \int_0^\infty d\alpha \ |\hat{f}(\omega(\vec{k})+\alpha)|^2 \ , \quad \omega(\vec{k}) = \sqrt{|\vec{k}|^2 + m^2} \ ,$$

which is certainly nonlocal!

Ansatz

We use then the following Ansatz for the quadratic form construction

$$\sum_{k=0}^{n}\frac{1}{\pi}\int_0^{+\infty} d\alpha\ C_{n-k}(\alpha)\left|\int dt\ \frac{:\varphi^k(t):}{k!}\ f(t)e^{i\alpha t}\right|^2 =$$

$$\sum_{l=0}^{n}\frac{1}{\pi}\int_0^{+\infty} d\alpha\ \frac{\alpha^{2l}}{(2l)!(2\pi)^{2l}}\iint dtdt' e^{i\alpha(t-t')}f(t)f(t')\ \frac{:\varphi^{n-l}(t)\varphi^{n-l}(t'):}{(n-l)!^2}$$

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The calculation is easy and the result is

$$C_k(\alpha) = \frac{1}{(2\pi)^{2k}} \underset{\substack{\sum_{j=1}^k \omega_j \le \alpha \\ \omega_j \ge m}}{\int \cdots \int} \omega_1 \cdots \omega_{k-1} \left(\omega_k - \sqrt{\omega_k^2 - m^2} \right) d\omega_1 \cdots d\omega_k \,,$$

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and we get positive quadratic forms for even Wicks' polynomials that are additive also for the massive case. We can further generalise these constructions to odd Wick's polynomials, for instance.

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From Haag duality for the von Neumann algebras $\mathfrak{A}(\mathcal{O})$, for a diamond $\mathcal{O}$, generated by elements $\mathsf{W}(\mathsf{g})$ with $\mathrm{supp}(\mathsf{g}) \subset \mathcal{O}$ one gets

Theorem

The selfadjoint extension $\overline{\mathsf{A}}(\mathsf{f})$ is affiliated with the von Neumann algebra $\mathfrak{A}(\mathcal{O})$ when $\mathrm{supp}(\mathsf{f}) \subset \mathcal{O}$

Some further results

I have yet to answer two of the open questions seen at the beginning.

Theorem

The selfadjoint extension $\overline{\mathsf{A}}(\mathsf{f})$ commutes with the Weyl operator $\mathsf{W}(\mathsf{g})$ if $\mathrm{supp}(\mathsf{f})$ is spacelike to $\mathrm{supp}(\mathsf{g})$

From Haag duality for the von Neumann algebras $\mathfrak{A}(\mathcal{O})$, for a diamond $\mathcal{O}$, generated by elements $\mathsf{W}(\mathsf{g})$ with $\mathrm{supp}(\mathsf{g}) \subset \mathcal{O}$ one gets

Theorem

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Eventually:

Theorem

Spectral projections for the selfadjoint $\overline{\mathsf{A}}(\mathsf{f})$ and $\overline{\mathsf{A}}(\mathsf{g})$ commute for spacelike separated f and g

Conclusions and Outlook

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R4 We have generalized Fewster-Ford-Roman quantum inequalities, at least on Minkowski spacetime.

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One may envisage applications to the following problems:

- A1 Spacetime coordinates for events
- A2 A version of resolvent algebra of Buchholz-Grundling tailored for positive and selfadjoint Wick's polynomials

Thank you!