Entanglement entropy and Lorentz invariance in QFT

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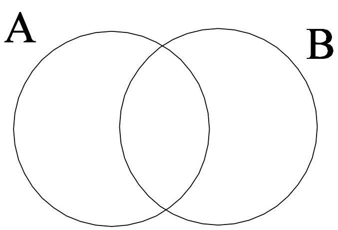
Strong subadditivity of entropy

$S(\rho) = -\mathrm{tr}(\rho \log \rho)$ von Neumann entropy

Strong subadditivity Lieb, Ruskai (1973)

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \longrightarrow S(\rho_{12}) + S(\rho_{23}) \geq S(\rho_2) + S(\rho_{123}) \qquad \rho_{12} = \mathrm{tr}_{H_3}(\rho)$$



$$S(A) + S(B) \geq S(A \cap B) + S(A \cup B)$$

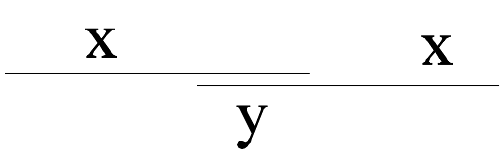
Additivity

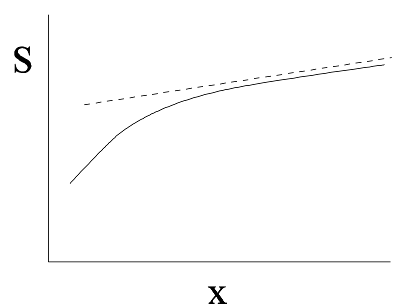
$$\rho_{AB} = \rho_A \otimes \rho_B \to S(AB) = S(A) + S(B)$$

Subadditivity

$$S(AB) \leq S(A) + S(B)$$

SSA + translational invariance (Robinson-Ruelle 1967): existence of entropy density

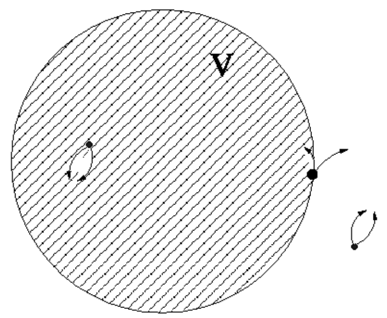




$$\lim_{x \to \infty} \frac{S(x)}{x} \to s$$

SSA + Lorentz symmetry: the vacuum state

Reduced density matrix $\rho_V = \mathrm{tr}_{-V} |0\rangle \langle 0| \longrightarrow$ S(V) entanglement entropy



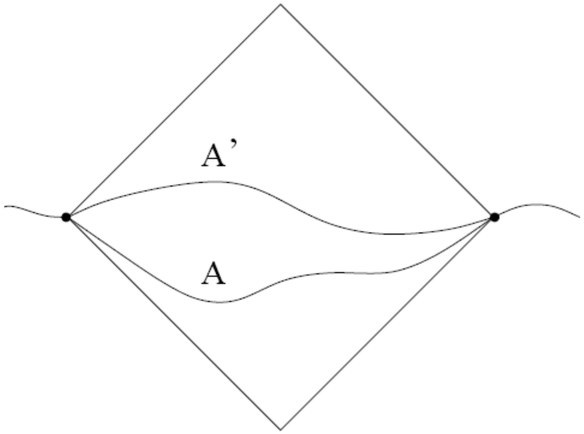
S(V) meassures the entropy in vacuum fluctuations

No boundary artificial conditions: S(V) property of the QFT

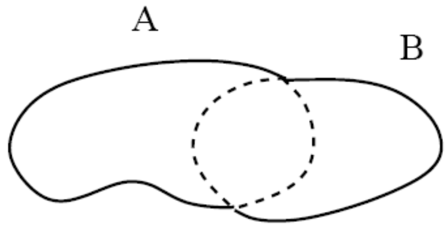
Region in space-time:

Causality $S(A) = S(A') \quad (\rho_A = \rho_{A'})$

S is a function of the "diamond shaped region"
of equivalently the region boundary (vacuum state
on an operator algebra)



Conditions for use of SSA:
"diagonalizing" Cauchy surface for A and B.
Boundaries must be spatial to each other



$$F(\mathcal{A}_A, \mathcal{A}_B) = S(\mathcal{A}_A) + S(\mathcal{A}_B) - S(\mathcal{A}_A \vee \mathcal{A}_B) - S(\mathcal{A}_A \wedge \mathcal{A}_B) \geq 0$$

$$\mathcal{C}(\mathcal{A}_A, \mathcal{A}_B) \equiv (\mathcal{A}_A \vee \mathcal{A}_B) \wedge (\mathcal{A}_A \vee \mathcal{A}'_B) \wedge (\mathcal{A}'_A \vee \mathcal{A}_B) \wedge (\mathcal{A}'_A \vee \mathcal{A}'_B)$$

$$\mathcal{C}(\mathcal{A}_A, \mathcal{A}_B) = Z \qquad ??$$

A geometric theorem

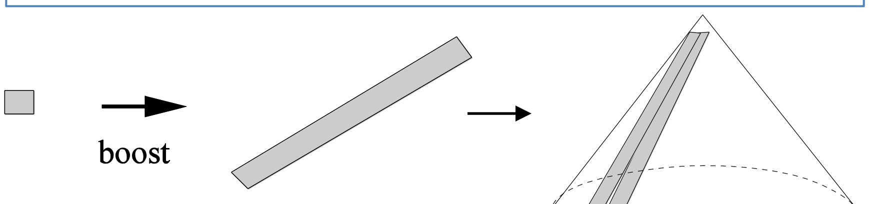
Entanglement entropy:

- Finite Regularized entropy?, BH entropy finite
- Poincare invariant
- Causality
- Strong subadditivity on Cauchy surfaces
- Positivity $S(A) \geq 0$

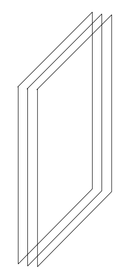
$$S(A) = s\ \ \mathrm{area}(A) + \alpha$$

Entropy linear in boundary area for relativistic polyhedrons

Poof: uses arbitrarily small and large distances!



$$S(A) \leq \mathrm{area}$$



+ translation invariance:

S ~ c N area

$$I(A,B) = S(A) + S(B) - S(A \cup B)$$

mutual information is zero

$$I(A,B) \geq \frac{1}{2} \frac{\left(\langle O_A O_B \rangle - \langle O_A \rangle \langle O_B \rangle\right)^2}{\|O_A\|^2 \|O_B\|^2}$$

is upper bound to correlations: no correlations: trivial theory

SSA very powerful combined with Lorentz symmetry
S divergent in QFT

Divergent and universal terms in EE

Structure of divergences

$$S(V) = \frac{g_{d-2}[\partial V]}{\epsilon^{d-2}} + \cdots + \frac{g_1[\partial V]}{\epsilon} + g_0[\partial V]\log(\epsilon) + S_0(V)$$

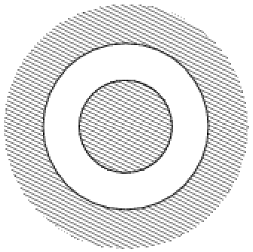
Area law: $S \sim \dfrac{R^{d-2}}{\epsilon^{d-2}}$

Divergences are local and extensive on the boundary of the region

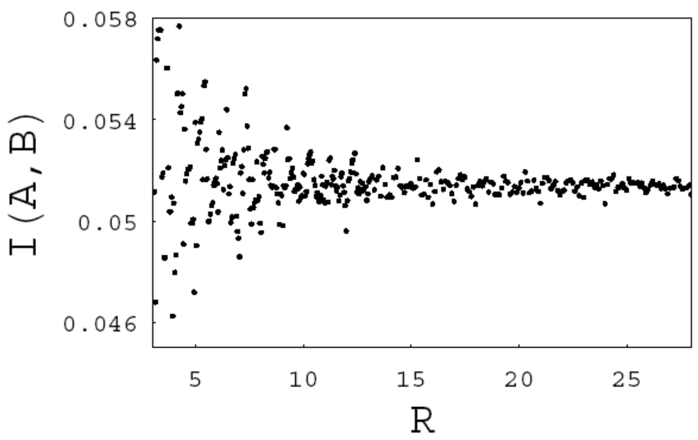
How to extract universal information?

One state two regions

Mutual information $I(A,B) = S(A) + S(B) - S(A \cup B)$



Divergences localized at the boundary cancel in I(A,B).



Universal information: Two states, one region

$$\Delta S(V) = S(V) - S_0(V)$$

(UV divergences independent of state)

Relative entropy $\quad S(\rho^1||\rho^0) = \mathrm{tr}(\rho^1 \log \rho^1 - \rho^1 \log \rho^0)$

$$S(\rho^1||\rho^0) = \Delta\langle H\rangle - \Delta S \qquad H = -\log \rho^0$$

(Half the modular Hamiltonian)

Application: Bekenstein bound $\quad S \leq \frac{2\pi R E}{\hbar c}$

Replace by $\longrightarrow \quad 2\pi \int_V x \,\langle \rho_E \rangle \geq S(V) - S_0(V)$

By Bisognano Whichman for the Rindler wedge V this is the positivity of relative entropy between vacuum and another state in half space

$$S(\rho^1||\rho^0) = \Delta\langle H\rangle - \Delta S$$

Araki formula gives a mathematical definition of

$$S(\rho^1||\rho^0)$$

Can $\Delta\langle H\rangle$ and ΔS be defined in a mathematically rigorous way?

Solve puzzles in original bound:

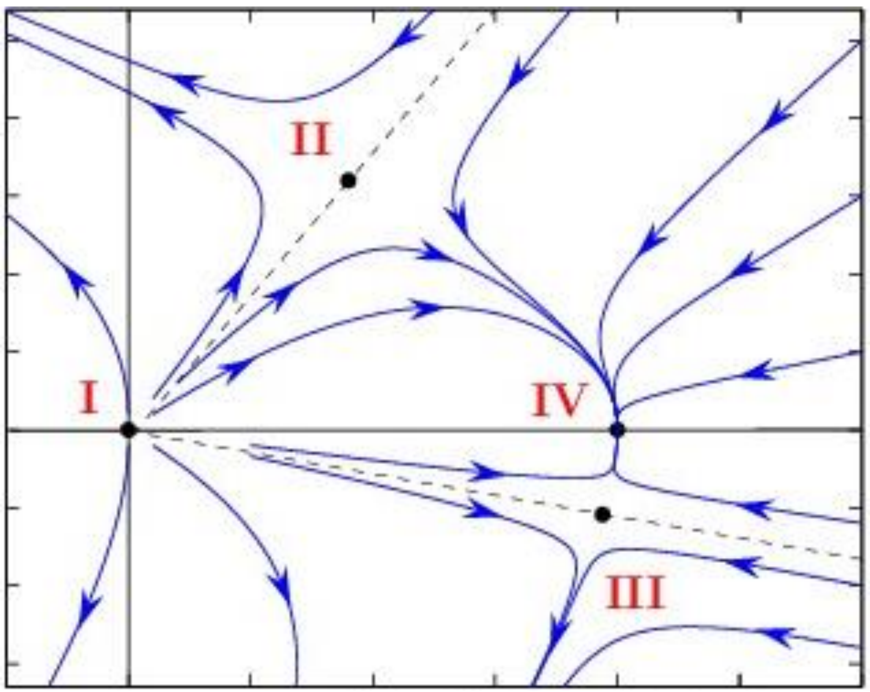
What is the meaning of R? Does it imply boundary conditions?

Was not the localized entropy divergent?

Species problem: What if we increase the number of particle species? (S increases, E is fixed)

The local energy can be negative while entropy is positive…

Renormalization group flow in the space of QFT



$$\tau \frac{dg_i}{d\tau} = \beta_i(\{g(\tau)\})$$

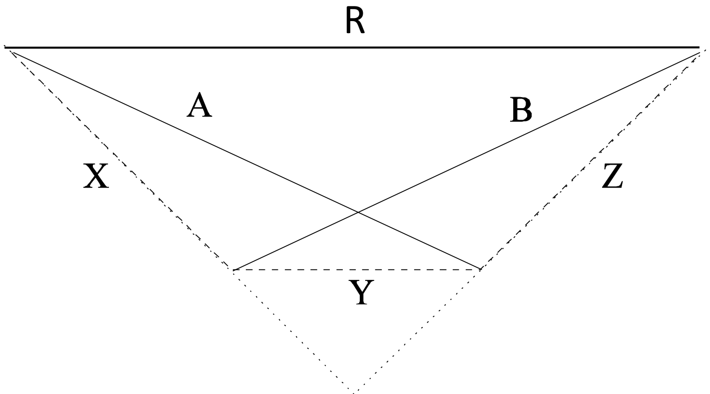
Change in the physics with scale through the change of coupling constants with the RG flow. At fixed points there is scale invariance: the theory looks the same at all scales. The RG flow Interpolates between UV (short distance) to IR (large distance) fix points.

Are there any general constraints on these RG flows?

Back to strong subadditivity

Entropic C-theorem from SSA in d=1+1

H.C., M. Huerta, 2004



$$S(XY)+S(YZ)\geq S(Y)+S(XYZ)$$

$$XY\equiv A\,,\qquad YZ\equiv B\,,\qquad XYZ\equiv R$$

$$2\,S(\sqrt{rR})\geq S(R)+S(r)\,.$$

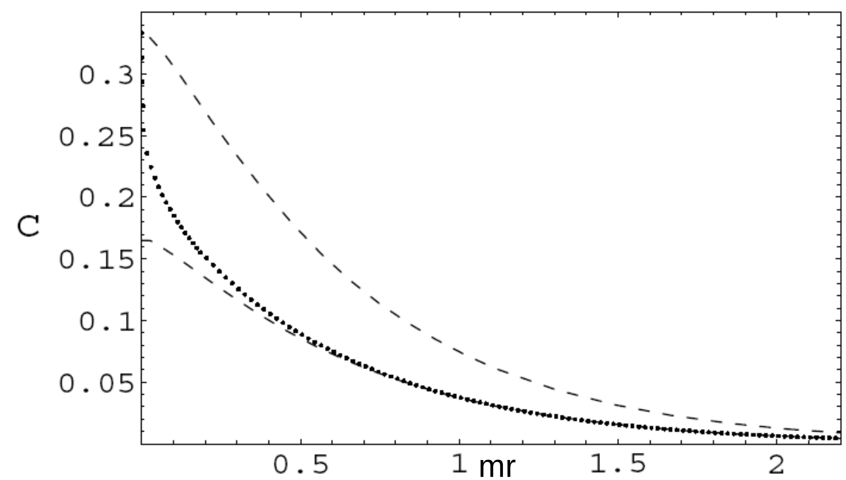
$$rS''(r)+S'(r)\leq 0\,.\qquad C(r)\;=\;rS'(r)\longrightarrow C'(r)\;\leq\;0$$

C(r) dimensionless, well defined, decreasing. At conformal points:

Holzey, Larsen, Wilczek (1992)
Calabrese, Cardy (2004)

$$S(r)=\frac{c}{3}\log(r/\epsilon)+c_0\longrightarrow$$ C(r)=c/3

The central charge of the uv conformal point must be larger than the central charge at the ir fixed point: the same result than Zamolodchikov c-theorem but different interpolating function



Again Lorentz symmetry and unitarity are used but in a different way

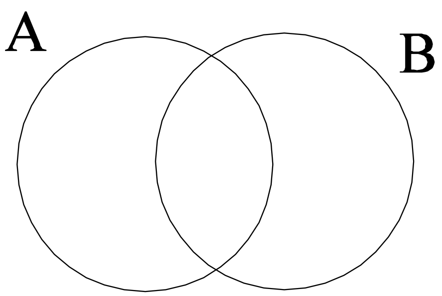
Entanglement entropy and irreversibility theorems

The EE of spheres at fix points of the RG contain the universal quantities in any dimension

$$S(r) = \mu_{d-2}\, r^{d-2} + \mu_{d-4}\, r^{d-4} + \cdots + \begin{cases} (-)^{\frac{d}{2}-1} 4\, A \,\log(R/\epsilon) & d \text{ even}\,. \\ (-)^{\frac{d-1}{2}} F & d \text{ odd}\,. \end{cases}$$

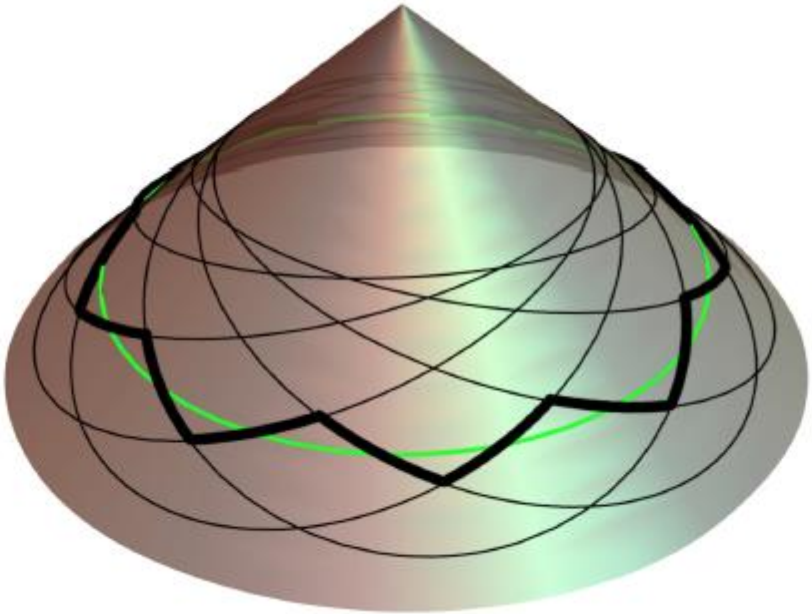
For even d: A is also the Log coefficient of the d-dimensional Euclidean sphere free energy or the Euler conformal anomaly. Cardy (1988) proposal. Proved d=2 Zamolodchikov (1986), d=4 Komargodski and Schwimmer (2011)	
For odd d: F is also the constant term of the d-dimensional Euclidean sphere free energy. No anomaly interpretation. Myers-Sinha (2010) and Klebanov et al (2011) proposals.	

Two problems: different shapes and new divergent contributions at intersection and union



H.C. and M. Huerta, 2012

SSA for multiple (boosted) spheres on the null cone



$$\sum_i S(X_i) \geq S(\cup_i X_i) + S(\cup_{\{ij\}}(X_i \cap X_j))$$
$$+S(\cup_{\{ijk\}}(X_i \cap X_j \cap X_k)) + \ldots + S(\cap_i X_i)\,.$$

$$S(\sqrt{rR}) \geq \frac{1}{N}\sum_{k=1}^{N} \tilde{S}_k \sim \int_r^R dl\, \beta(l)\tilde{S}(l)$$

$$\beta(l) = \frac{2^{d-3}\Gamma[(d-1)/2]}{\sqrt{\pi}\Gamma[(d-2)/2]}\,\frac{(rR)^{\frac{d-2}{2}}\left((l-r)(R-l)\right)^{\frac{d-4}{2}}}{l^{d-2}(R-r)^{d-3}}$$

Large N, small wiggle limit

IF we can replace wiggly spheres by spheres

$$rS''(r) - (d-3)S'(r) \leqslant 0$$

d=3: $S''(r) \leqslant 0$ $S_{FP}^{d=3}(r) = \mu_1 r - F$

$$\left\{ \begin{array}{lll} F(r) = rS'(r) - S(r) \quad \text{decreases} & & \text{F-theorem} \\ (S'(r))' \leqslant 0 & \Delta\mu_1 \leqslant 0 & \text{«area theorem»} \end{array} \right.$$

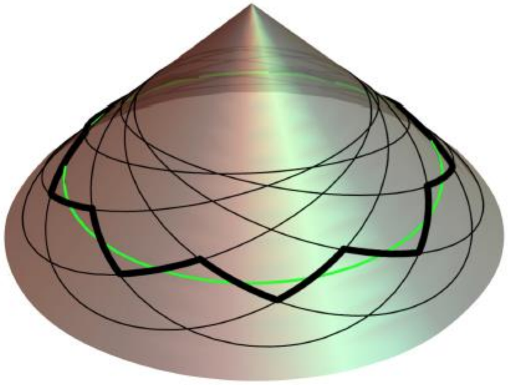
~ G grows towards the IR

F is the topological EE for a topological IR fix point. Very non local.

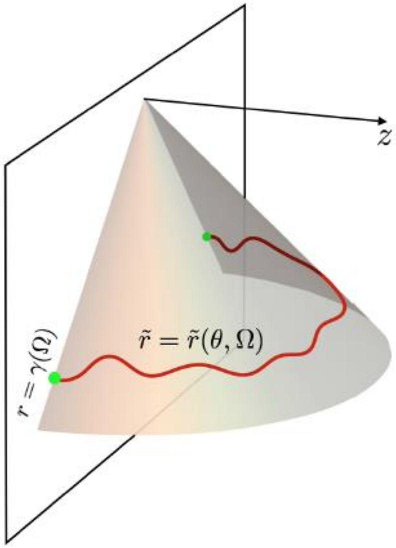
d=4: $rS''(r) - S'(r) \leqslant 0$

For a fix point $S(r) = \mu_2 r^2 - 4A\log(r/\epsilon) \longrightarrow rS''(r) - S'(r) = \frac{8A}{r} > 0$

The naive inequality is incorrect! What is going on?



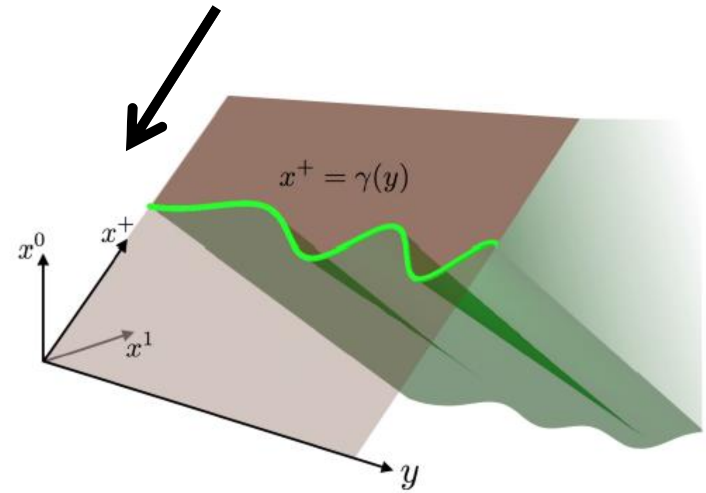
We cannot replace wiggly spheres by spheres: there is a finite contribution of the wiggles that do not vanish for wiggle size going to zero.



Holographic result:

$$A = \int d\Omega \left(\frac{r(\Omega)^2}{2\epsilon^2} - \frac{1}{4} r \nabla^2_\Omega r^{-1} - \frac{1}{4} - \frac{1}{2}\log(2r(\Omega)/\epsilon) \right)$$

How can this be repaired? Markov property of vacuum state (H.C, E. Teste, G. Torroba (2017), (2018))



boost $x^+ \to \lambda x^+$

$$S(\gamma) = S(\lambda\gamma)$$

Then all entropies on the null plane are equal ! (with Lorentz invariant regularization)

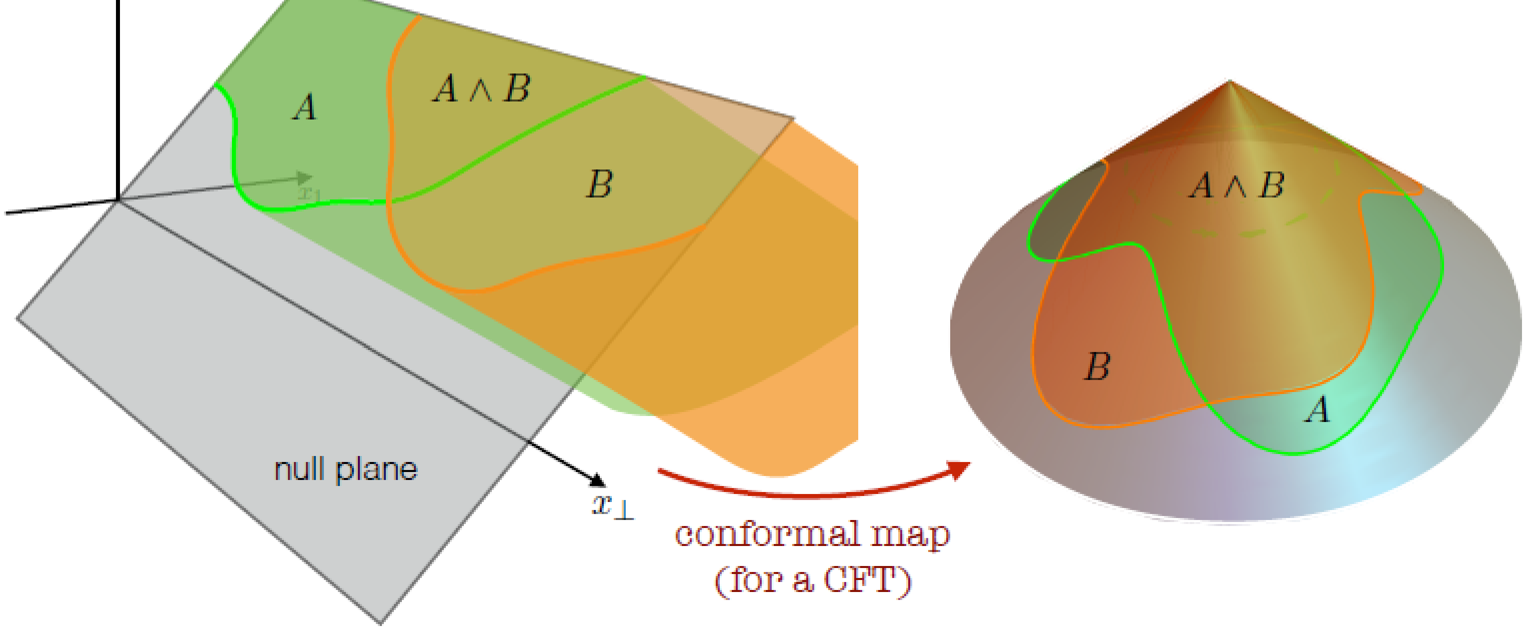
Then

$$S(A) + S(B) - S(A \cap B) - S(A \cup B) = 0$$

Markov property: This is now a cutoff independent statement

The statement for the vacuum

$$S(\rho_A) + S(\rho_B) = S(\rho_{A \vee B}) + S(\rho_{A \wedge B})$$



$$S(A) + S(B) = S(A \wedge B) + S(A \vee B)$$

for the vacuum of any QFT

$$S(A) + S(B) = S(A \wedge B) + S(A \vee B)$$

for the vacuum of any CFT

Markov property equivalent to the same property for the modular Hamiltonians

$$S(A) + S(B) - S(A\cap B) - S(A\cup B) = 0 \Leftrightarrow H_A + H_B - H_{A\cap B} - H_{A\cup B} = 0$$

H is local «null line by null line»

Generalization of Bisognano Whichmann on the null plane for QFT

$$\hat{H}_\gamma = 2\pi \int d^{d-2}x^\perp \int_{-\infty}^{\infty} d\lambda\, (\lambda - \gamma(x^\perp)) T_{\lambda\lambda}(\lambda, x^\perp)$$

And of Hislop and Longo on the Null cone for a CFT

$$H_\gamma = 2\pi \int d\Omega\, \gamma^{d-3}(\Omega) \int_0^{\gamma(\Omega)} d\lambda\, \lambda\, (\gamma(\Omega) - \lambda)\, T_{\lambda\lambda}$$

One can show that they satisfy the correct algebra $[\hat{H}_{\gamma_1}, \hat{H}_{\gamma_2}] = 2\pi i(\hat{H}_{\gamma_1} - \hat{H}_{\gamma_2})$

And positivity (ANEC) $\int d\lambda\, T_{\lambda\lambda}(\lambda, x^\perp) > 0$

(Faulkner et al (2016), Hartman et al (2017))

Consistent with half sided modular inclusions (among these regions)

In general the unitary modular flows act locally only on the null plane, and non locally outside, excepting for the case of wedges or spheres in a CFT, which correspond to Noether charges of geometric symmetries.

Markov property: roughy speacking vacuum is a product state along null generators.
A result in the opposite direction of Reeh-Slieder theorem.

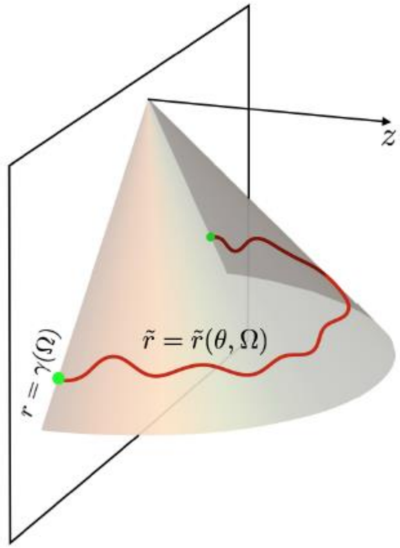
The entropies on the null cone for a CFT are given by local functional.

Lorentz invariance on the cone -> conformal invariance on the sphere

Universal form for the entropies given by conformal invariant action for dilaton $\phi(y) = \log(\gamma(y)/\epsilon)$

Recall holographic result for d=4:

$$\mathcal{A} = L^3 \int d^2\Omega \left\{ \frac{1}{2}\frac{\gamma^2}{\epsilon^2} - \frac{1}{2}\log\frac{\gamma}{\epsilon} - \frac{1}{4}\left(\frac{\nabla_\Omega \gamma}{\gamma}\right)^2 + \mathcal{O}(\epsilon^0) \right\}$$



Wess-Zumino action

Returning to the A theorem in d=4. How to transform the wiggly spheres into spheres?
Use Markov property

$$S_{\rho^1}(r) \to \ \Delta S(r) = S_{\rho^1}(r) - S_{\rho^0}(r)$$

Get rid of the wiggles using the entropy differences with the UV-CFT: wiggles are short distance and mass corrections to the finite wiggle term will vanish for large N.

$$\Delta S(\sqrt{rR}) \geq \frac{1}{N}\sum_{k=1}^{N}\Delta S_k \approx \int_r^R dl\,\beta(l)\,\Delta S(l) \qquad \longrightarrow \qquad r\,\Delta S''(r) - (d-3)\,\Delta S'(r) \leqslant 0$$

d=4: $r\Delta S''(r) - \Delta S'(r) \leqslant 0$

$$S_{\rho^0}(r) = \mu_2^0\, r^2 - 4A_{UV}\log(r/\epsilon)$$

Evaluating at the IR we get $A_{IR} \leqslant A_{UV}$ $\longrightarrow$ A-theorem

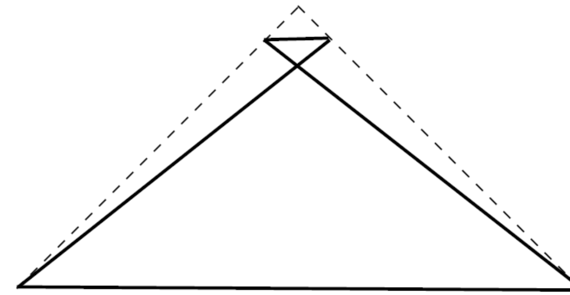
For d>4 we get $\Delta\mu_{d-4} \geqslant 0$

SSA not enough to say something about the universal terms. More inequalities?

All c/F/a theorems in the same footing

SSA, Lorentz invariance, and causality, allows us to compare entropies of very different sizes

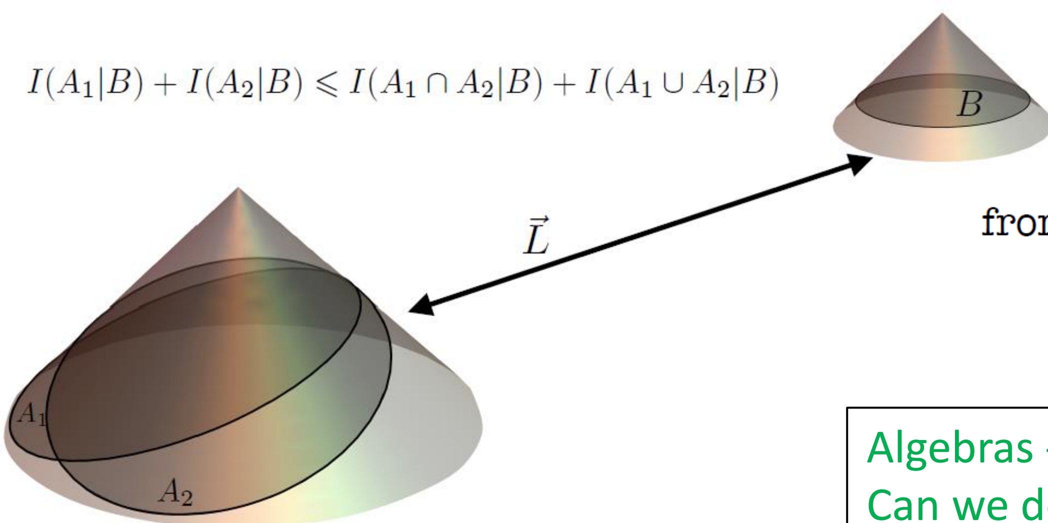
Why the cone construction: Markov property on the cone ensures SSA saturates for a CFT. Otherwise we would have no chance to obtain useful information about the RG flow. The null cone or plane: only case where Markov property holds in vacuum CFT



Unitarity bound for mutual information

Mutual information cannot decay slower than $L^{-2(d-2)}$

$$I(A_1|B)+I(A_2|B)\leqslant I(A_1\cap A_2|B)+I(A_1\cup A_2|B)$$



from [Cardy, 13] $I(A_i|B)\sim\dfrac{F(r_{A_i})F(r_B)}{L^{4\Delta}}$

$\longrightarrow$ $\Delta\geqslant\dfrac{d-2}{2}$

Algebras -> numbers
Can we define a QFT by its vacuum EE?
What other axioms are needed?

For higher dimensions we get two inequalities:

The first is the decrease of the quantity $\frac{\Delta S'(r)}{(d-2)r^{d-3}}$

This gives the decrease of the area coefficient with size for any d. The area coefficient gets finitely renormalized only for $\Delta < (d+2)/2$

The second follows evaluating the IR value of $r\,\Delta S''(r) - (d-3)\,\Delta S'(r) \leqslant 0$

that gives $\lim_{r\to\infty} \Delta\mu_{d-4} \geqslant 0$

This is finitely renormalized for $\Delta < (d+4)/2$

Could it be that $\Delta\mu_{d-k}$ has sign $(-)^{k/2}$ for any d and k ? (and this will give us the c-theorem in more dimensions)

Write it in terms of the relative entropy between different vacua using $H_A + H_B - H_{A\cap B} - H_{A\cup B} = 0$

Work still needed to make EE a better understood tool in QFT
In particular better understanding of EE for free (and asymptotically free) gauge theories needed

Markov chain

Classically

$$p(x|y,z) = p(x|y)$$

$$\downarrow$$

$$p(x,y,z) = \frac{p(x,y)p(y,z)}{p(y)}$$

$$\downarrow$$

$$\log p(x,y) + \log p(y,z) = \log p(x,y,z) + \log p(y)$$

$$\downarrow$$

$$S(p(x,y)) + S(p(y,z)) = S(p(x,y,z)) + S(p(y))$$

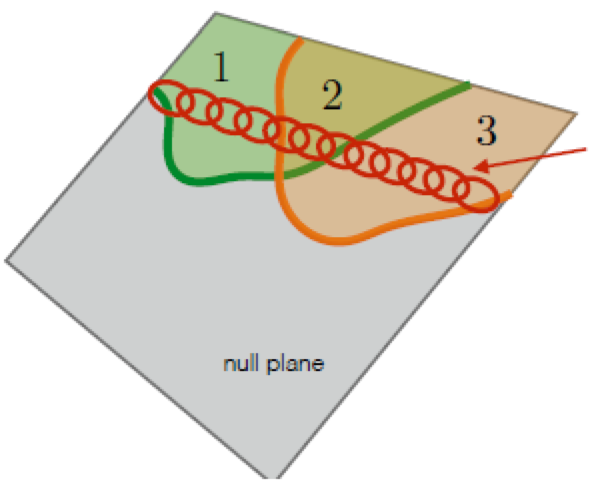
The condition $S(\rho_{12}) + S(\rho_{23}) = S(\rho_{123}) + S(\rho_2)$ is strong enough to fix great part of the structure of the state

$\downarrow$ [Hayden,Jozsa, Petz,Winter: 04]

$\exists$ a decomposition of $\mathcal{H}_2 = \oplus_k \mathcal{H}_{2L}^k \otimes \mathcal{H}_{2R}^k$

such that $\rho_{1\cancel{2}3} = \sum_k p_k \; \rho_{1\cancel{L}}^k \otimes \rho_{\cancel{R}3}^k$ separable

The entanglement structure of a Markov state is like a chain: If we cut a link (take trace on the intersection algebra) the state becomes separable



the vacuum is like this (with respect to null cut subalgefbras)

Holographic check of saturation of SSA. The null plane

$$ds^2 = \frac{L^2}{z^2}\left(f^2(z)dz^2 - dx^+dx^- + d\vec{y}^2\right)$$

boundary

$$\begin{aligned} x^+ &= \gamma(\vec{y}) \\ x^- &= 0\,. \end{aligned}$$

Minimal surface solution (linear equation!)

$$x^- = 0$$

$$\nabla_y^2 x^+ + \frac{1}{f^2}\left(\frac{\partial^2 x^+}{\partial z^2} - \left(\frac{f'}{f} + \frac{d-1}{z}\right)\frac{\partial x^+}{\partial z}\right) = 0$$

For pure AdS

$$\begin{aligned} x^+(z,y) &= \frac{2^{1-d/2}}{\Gamma[d/2]}\int d^{d-2}k\; a_{\vec{k}}\; e^{i\vec{k}\cdot\vec{y}}\;(|\vec{k}|z)^{d/2}\;K_{d/2}(|\vec{k}|z) \\ a_{\vec{k}} &= \int \frac{d^{d-2}y}{(2\pi)^{d-2}}\;e^{-i\vec{k}\cdot\vec{y}}\;\gamma(\vec{y})\,. \end{aligned}$$

In any case the induced metric on the surface does not depend on the shape and all entropies are equal, SSA saturates

$$ds^2|_{\mathcal{M}} = \frac{L^2}{z^2}\left(f^2(z)dz^2 + d\vec{y}^2\right)$$

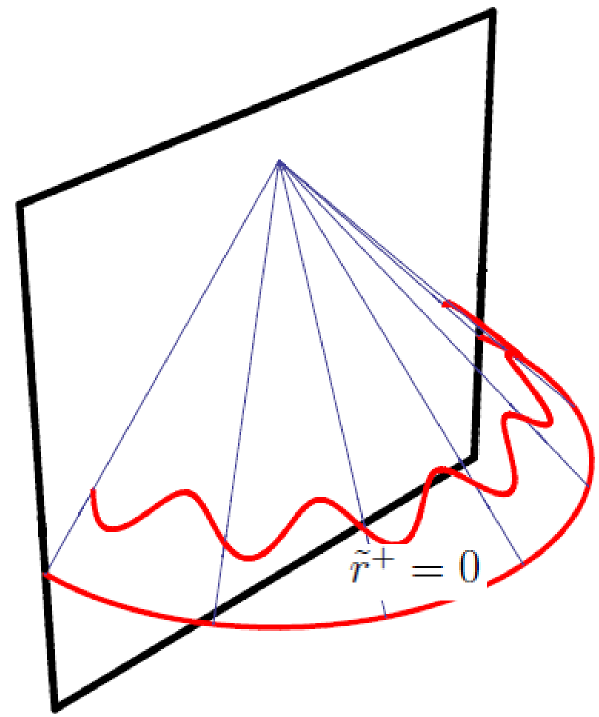
Holographic saturation of SSA on the cone

$$\tilde{r} = \sqrt{r^2 + z^2} \qquad \tilde{r}^{\pm} = \tilde{r} \pm t$$

$$ds^2 = \frac{d\tilde{r}^+ d\tilde{r}^- + \tilde{r}^2 d\tilde{\Omega}^2}{\tilde{r}^2 \sin^2(\theta)}$$

$$d\tilde{\Omega}^2 = d\theta^2 + \cos^2(\theta) d\Omega^2$$

$$\left(\frac{\partial^2}{\partial\theta^2} - ((d-2)\tan\theta + (d-1)\cot\theta)\frac{\partial}{\partial\theta} + \frac{1}{\cos^2\theta}\nabla^2_{\Omega}\right)(\tilde{r}^-)^{-1} = 0$$



The area again does not depend on the surface shape…

$$ds^2 = \frac{d\tilde{\Omega}^2}{\sin^2(\theta)} = \frac{d\theta^2 + \cos(\theta)^2 d\Omega^2}{\sin^2\theta} \qquad A = \int d\Omega \int_{\beta(\Omega)}^{\pi/2} d\theta \, \frac{\cos(\theta)^{d-2}}{\sin(\theta)^{d-1}} \, .$$

except for the position of the cutoff $\quad z = \tilde{r}\sin(\theta) = \epsilon$

d=3:

$$A = \int d\varphi \, \frac{r(\varphi)}{\epsilon} - 2\pi$$

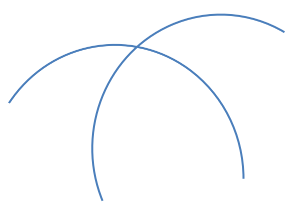
Trivially saturates SSA. All surfaces have the same constant term F

d=4:

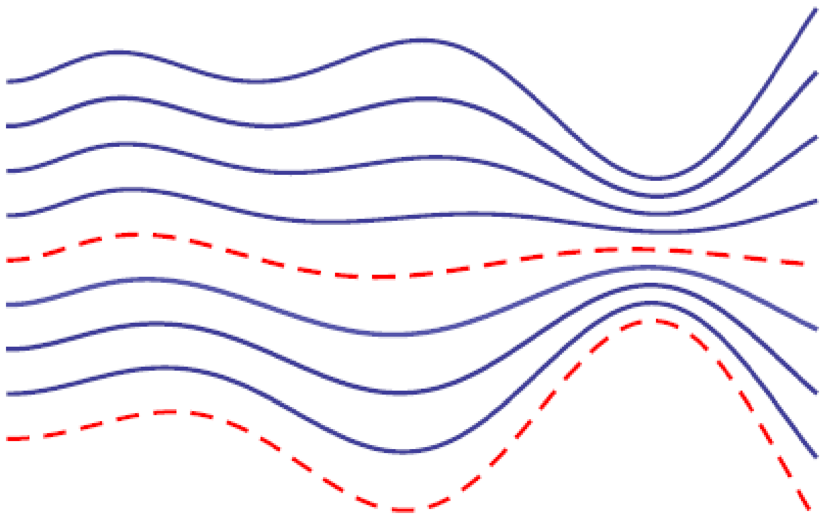
$$A = \int d\Omega \left(\frac{r(\Omega)^2}{2\epsilon^2} - \frac{1}{4} r \nabla^2_\Omega r^{-1} - \frac{1}{4} - \frac{1}{2} \log(2r(\Omega)/\epsilon) \right)$$

All surfaces have the same log divergence (proportional to the A anomaly). This is because there is only one extrinsic curvature in Solodukhin's formula, proportional to the Euler density of the surface. The finite term (Lorentz invariant) is not a curvature term, even if it is an integral on the boundary of the region.

Saturation of SSA holds because A is an integral of a local function on the boundary. Cusp terms cancell each other between the intersection and the union:



$$\int_{P_i} d\Omega \, r \, \nabla^2_\Omega r^{-1} = \int_{P_i} d\Omega \, \frac{\nabla_\Omega r \cdot \nabla_\Omega r}{r^2} - \int_{\partial P_i} dl \, \eta \cdot \frac{\nabla_\Omega r}{r}$$



On the null plane we define a modular flow is "standard" if it moves the local algebras in a geometric way according to (this is for example the action of modular flows of wedges)

$$U_{\gamma_1}(-s)\gamma_2 U_{\gamma_1}(s) = e^{2\pi s}(\gamma_2 - \gamma_1) + \gamma_1$$

These are like dilatations on each null ray

In this case for modular inclusions the modular translations act geometrically like translations on each ray

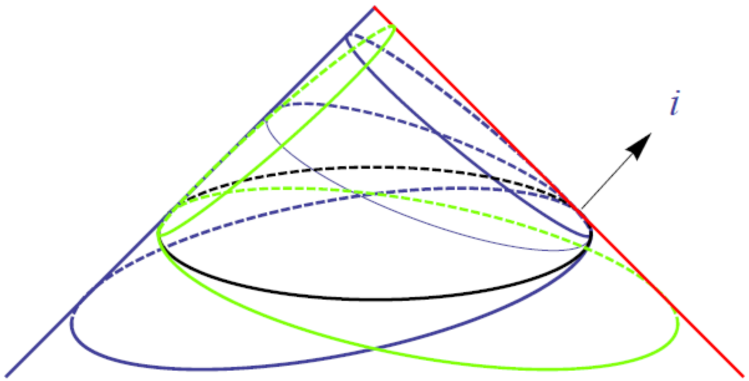
$$T_{\gamma_{\tau_1},\gamma_{\tau_2}}(-\tau)\gamma T_{\gamma_{\tau_1},\gamma_{\tau_2}}(\tau) = \gamma + 2\pi\tau(\gamma_{\tau_2} - \gamma_{\tau_1})$$

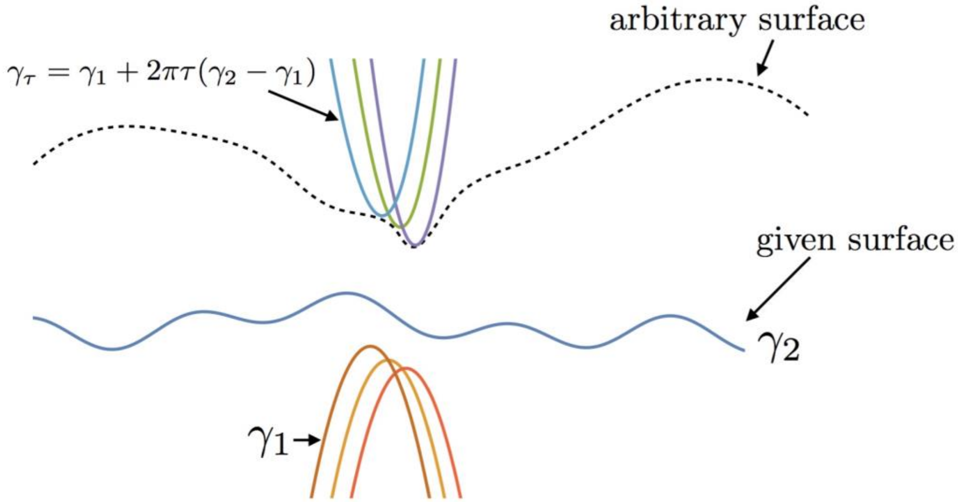
If a modular flow moves in a standard way two regions it will also move the intersection and the union

$$\gamma_2 \cap \gamma_3 = \theta(\gamma_2 - \gamma_3)\gamma_3 + \theta(\gamma_3 - \gamma_2)\gamma_2$$

$$\gamma_2 \cup \gamma_3 = \theta(\gamma_2 - \gamma_3)\gamma_2 + \theta(\gamma_3 - \gamma_2)\gamma_3$$

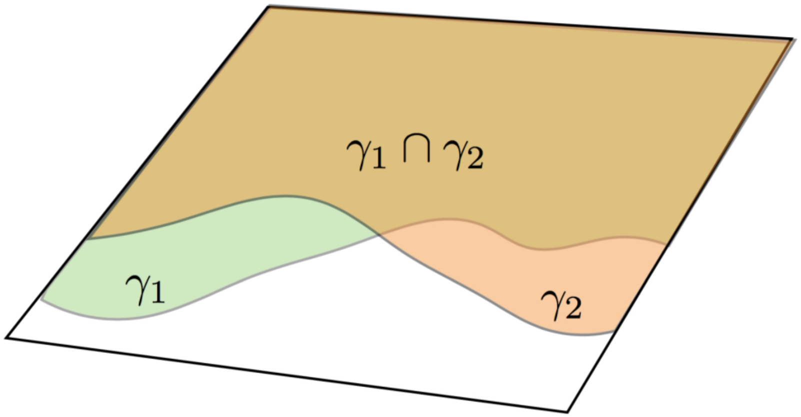
The modular flows of wedges move regions in the standard way. For a CFT we have also parabolas that are conformal transformations of spheres with standard geometric flows





Using unions of modular translated parabolas one can show any region will move in a standard way any other region above it

Then for two intersecting regions
We have for the action of the modular translations on another region above them



$$T_{\gamma_1,\gamma_1\cap\gamma_2}(-\tau)\gamma_3 T_{\gamma_1,\gamma_1\cap\gamma_2}(\tau) = \gamma_3 + 2\pi\tau(\gamma_1\cap\gamma_2 - \gamma_1)$$
$$= \gamma_3 + 2\pi\tau(\gamma_2 - \gamma_1\cup\gamma_2) = T_{\gamma_1\cup\gamma_2,\gamma_2}(-\tau)\gamma_3 T_{\gamma_1\cup\gamma_2,\gamma_2}(\tau)\,.$$

The question is if this means the two modular translations are identical operators. For this we use:

> If a one parameter unitary group keeping the vacuum invariant and with positive generator translates a family of nested algebras into themselves, this group is unique

The equality for the modular translations implies the Markov property of modular Hamiltonians

$$\hat{H}_{\gamma_1} - \hat{H}_{\gamma_1\cap\gamma_2} = \hat{H}_{\gamma_1\cup\gamma_2} - \hat{H}_{\gamma_2}$$

$$
\begin{aligned}
P_{x^\perp} &= \int d\lambda\, T_{\lambda\lambda}(\lambda, x^\perp)\,, \\
K_{x^\perp} &= \int d\lambda\, \lambda\, T_{\lambda\lambda}(\lambda, x^\perp)\,.
\end{aligned}
$$

$$[K_{x^\perp}, P_{x^\perp\prime}] = \delta(x^\perp - x^{\perp\prime}) O_0(x^\perp) + \partial_i \delta(x^\perp - x^{\perp\prime}) O_1^i(x^\perp) + \ldots$$

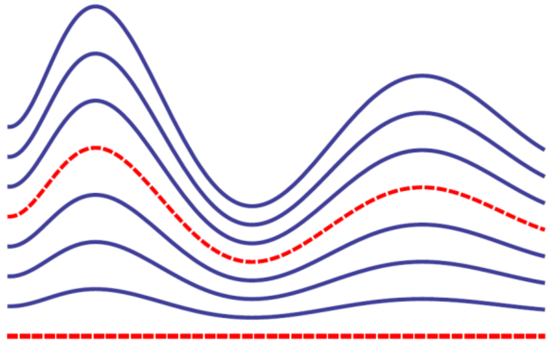
$$O_0 = \int d\lambda\, \Phi(\lambda)\,.$$

twist $\tau = \Delta - s = d - 2.$

$$[K_{x^\perp}, P_{x^\perp\prime}] = -i P_{x^\perp} \delta(x^\perp - x^{\perp\prime}) \longrightarrow [\hat{H}_{\gamma_1}, \hat{H}_{\gamma_2}] = 2\pi i(\hat{H}_{\gamma_1} - \hat{H}_{\gamma_2})$$

for $$\hat{H}_\gamma = 2\pi \int d^{d-2}x^\perp \int_{-\infty}^{\infty} d\lambda\, (\lambda - \gamma(x^\perp)) T_{\lambda\lambda}(\lambda, x^\perp)$$

On the null plane a Rindler wedge and a region above it are half-sided



$$H_W = 2\pi K_1$$

$$T_{W,\gamma}(-\tau)\,\gamma\,T_{W,\gamma}(\tau) = (1+2\pi\tau)\gamma$$

We need to compute these translation generators. Let us define

$$P_{x^\perp} = \int d\lambda\, T_{\lambda\lambda}(\lambda, x^\perp)\,,$$

From the ANEC
this is a positive operator

Faulkner, Leigh, Parrikar, Wang (2016)
Hartman, Kundu, Tajdini (2017)

$$\int dx^\perp \langle 0|P_{x^\perp}|0\rangle = 0 \longrightarrow P_{x^\perp}|0\rangle = 0$$

These "translations" are positive and annihilate the vacuum

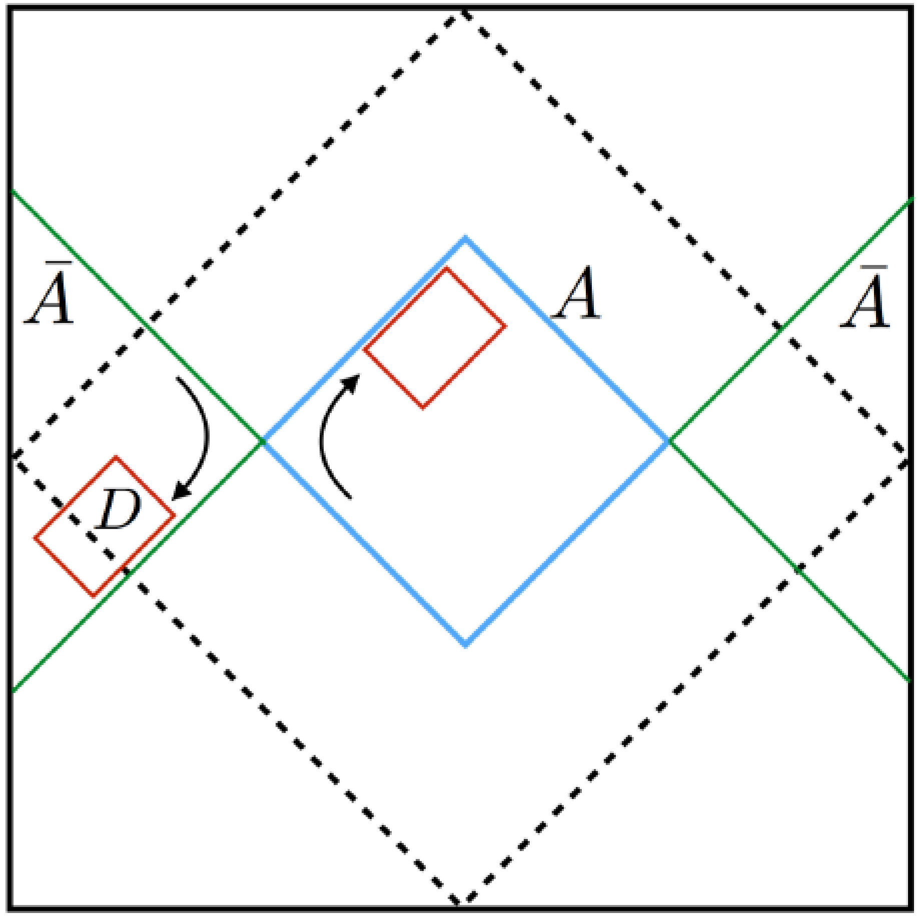
$$P_{2\pi\gamma} = \int d^{d-2}x^\perp\,(2\pi\gamma(x^\perp))\int d\lambda\, T_{\lambda\lambda}(\lambda, x^\perp)$$

They move operators in the null surface locally:
move algebras geometrically in the same way as $T_{W,\gamma}(\tau)$

$$e^{i\tau P_{2\pi\gamma}}\gamma e^{-i\tau P_{2\pi\gamma}} = (1+2\pi\tau)\gamma$$

A one parameter unitary group keeping the vacuum invariant and with positive generator translating a family of nested algebras into themselves is unique

$$G_{W,\gamma} = P_{2\pi\gamma} \longrightarrow \hat{H}_\gamma = \hat{H}_W - P_{2\pi\gamma}$$



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