Entanglement Measures and Modular Theory

S. Hollands

UNIVERSITÄT LEIPZIG

mostly based on joint work with K Sanders

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What is entanglement?

Entanglement

Entanglement concerns subsystems (usually two, called A and B) of an ambient system. Roughly, one asks how much "information" one can extract about the state of the total system by performing **separately local, coordinated operations** in A and B.

Basic setup

Abstractly, the typical setup for (bipartite) entanglement is as follows:

Setup

Two commuting v. Neumann algebras $\mathfrak{A}_A, \mathfrak{A}_B$ defined on common Hilbert space $\mathcal{H}$ with unitary identification $\mathfrak{A}_A \vee \mathfrak{A}_B \cong \mathfrak{A}_A \otimes \mathfrak{A}_B$.

Example 1: $\mathfrak{A}_A = M_n(\mathbb{C}) = \mathfrak{A}_B$ realized on Hilbert space $\mathcal{H} = \mathbb{C}^n \otimes \mathbb{C}^n$ with standard inner product. States of the system correspond to vectors or density matrices on $\mathcal{H}$. ("Type I case")

Example 2: $\mathfrak{A}_A = L^\infty(X), \mathfrak{A}_B = L^\infty(Y)$: Classical situation. Probability distributions $p \in L^1(X \times Y)$ give states.

Example 3: Let $A, B \subset \mathbb{R}^d$, and $\mathfrak{A}_A, \mathfrak{A}_B$ the algebras of observables of a quantum field theory localized in corresponding "causal diamonds" $O_A, O_B \subset \mathbb{R}^{d,1}$. ("Type III case")

Localization in QFT

In QFT, systems are tied to spacetime localization, e.g. system A

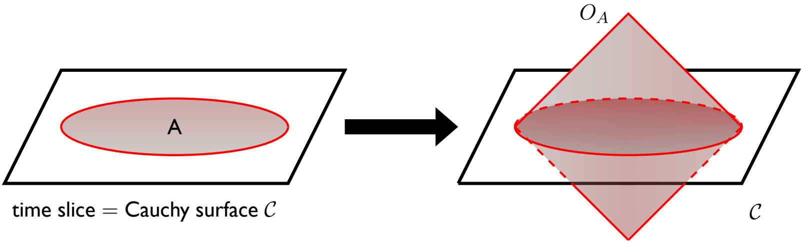


Figure: Causal diamond O_A associated with A.

Set of observables measurable within O_A is an algebra $\mathfrak{A}_A =$ "quantum fields localized at points in O_A". If A and B are regions on time slice (Einstein causality) [Haag, Kastler 1964]

$$[\mathfrak{A}_A, \mathfrak{A}_B] = \{0\} \ .$$

The algebra of all observables in A and B is called $\mathfrak{A}_A \vee \mathfrak{A}_B =$ v. Neumann algebra generated by $\mathfrak{A}_A$ and $\mathfrak{A}_B$.

What is entanglement?

Abstract version of states:

Given an abstract v. Neumann algebra $\mathfrak{A}_A \vee \mathfrak{A}_B \cong \mathfrak{A}_A \otimes \mathfrak{A}_B$, states are positive normalized, normal linear functionals ω on $\mathfrak{A}_A \otimes \mathfrak{A}_B$.

Example 1: $\mathfrak{A}_A = M_n(\mathbb{C}) = \mathfrak{A}_B$. All states of form $\omega(a) = \mathrm{Tr}_{\mathcal{H}}(\rho_\omega a)$ for density matrix ρ_ω on $\mathcal{H} = \mathbb{C}^n \otimes \mathbb{C}^n$.

Separable states:

A state is called **separable** if it is a finite sum of the form $\omega = \sum \omega_{Ai} \otimes \omega_{Bi}$ where $\omega_{Ai} \otimes \omega_{Bi}(a \otimes b) = \omega_{Ai}(a)\omega_{Bi}(b)$ is normal (product state).

Example 2: $\mathfrak{A}_A = L^\infty(X), \mathfrak{A}_B = L^\infty(Y)$: Basically every state $p \in L^1(X \times Y)$ is a limit of separable states.

Remark: Normal product states will sometimes not exist (see below)!

What is entanglement?

Example 2 motivates:

Entangled states

A state is called **entangled** if it is **not** in the norm closure of separable states.

Example 1: $\mathfrak{A}_A = M_2(\mathbb{C}) = \mathfrak{A}_B$ spin-1/2 systems, Bell state $\rho = |\Omega\rangle\langle\Omega|$

$$|\Omega\rangle = 2^{-1/2}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle).$$

is (maximally) entangled.

Example 2: Type I_n: $\mathfrak{A}_A = M_n(\mathbb{C}) = \mathfrak{A}_B$:

$$|\Omega\rangle = n^{-1/2} \sum_j |j\rangle \otimes |j\rangle$$

Example 3: Type I_∞:

$$|\Omega\rangle = Z_\beta^{-1/2} \sum_j e^{-\beta E_j/2} |j\rangle \otimes |j\rangle \quad (\rightarrow \text{KMS condition})$$

Situation in QFT

Unfortunately [Buchholz, Wichmann 1986, Buchholz, D'Antoni, Fredenhagen 1987, Doplicher, Longo 1984, ... :

$$[\mathfrak{A}_A, \mathfrak{A}_B] = \{0\} \quad \text{does not always imply} \quad \mathfrak{A}_A \vee \mathfrak{A}_B \cong \mathfrak{A}_A \otimes \mathfrak{A}_B \ .$$

This will happen due to boundary effects if A and B touch each other (algebras are of type III_1 in Connes classification):

Basic conclusion

a) If A and B touch, then there are no (normal) product states, so no separable states, and no basis for discussing entanglement!

b) If A and B do not touch, then there are no pure states (without firewalls)!

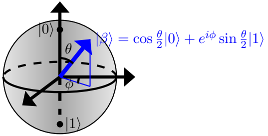
Therefore, if we want to discuss entanglement, we **must** leave a safety corridor between A and B, and we **must** accept b).

What to do with entangled states?

Now and then:

Then: EPR say (1935) Entanglement = "spooky action-at-a-distance"
Now: Entanglement = resource for doing new things!

Example: Teleportation of a state $|\beta\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$ from A to B. [Bennett, Brassard, Crepeau, Jozsa, Perez, Wootters 1993].



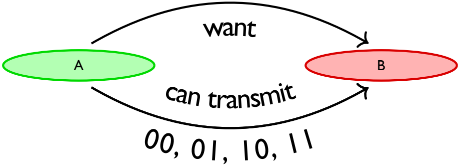


Figure: Teleportation of one q-bit.

Quantum teleportation

Basic lessons:

- To teleport **one** "q-bit" $|\beta\rangle$ need **one** Bell-pair entangled across A and B! $\Rightarrow$ For lots of q-bits need lots of entanglement.
- Teleportation "protocol" consists of **sequence of separable operations** and classical communications (see below). These "use up" the entanglement of the original Bell-pair.

When is a state more entangled than another?

In type I_n situation, a channel is:

- Time evolution/gate: unitary transformation: $\mathcal{F}(a) = UaU^*$
- Ancillae: n copies of system: $\mathcal{F}(a) = 1_{\mathbb{C}^n} \otimes a$
- v. Neumann measurement: $\mathcal{F}(a) = PaP$, where $P : \mathcal{H} \to \mathcal{H}'$ projection
- Arbitrary combinations $=$ completely positive (cp) maps [Stinespring 1955]

In general case, channel is a normalized $\mathcal{F}(1) = 1$, normal, cp map. ($\mathcal{F} : \mathfrak{M}_1 \to \mathfrak{M}_2$ cp $\Leftrightarrow 1_{\mathbb{C}^2} \otimes \mathcal{F}$ positive.) **Bipartite** system:

Separable operations ("$=$ channels $+$ classical communications"):

Normalized sum of product channels, $\sum \mathcal{F}_A \otimes \mathcal{F}_B$ acting on operator algebra $\mathfrak{A}_A \otimes \mathfrak{A}_B$

Entanglement measures

Basic properties:

Definition of entanglement measure E:

A state functional $\omega \mapsto E(\omega)$ on $\mathfrak{A}_A \otimes \mathfrak{A}_B$ such that

- (e1) $E(\omega) \geq 0$.
- (e2) $E(\omega) = 0 \Leftrightarrow \omega$ separable.
- (e3) Convexity $\sum p_i E(\omega_i) \geq E(\sum p_i \omega_i)$.
- (e4) No increase "on average" under separable operations:

  $$\sum_i p_i E(\tfrac{1}{p_i} \mathcal{F}_i^* \omega) \leq E(\omega)$$

  for all states ω (NB: $p_i = \mathcal{F}_i^* \omega(1) =$ probability that i-th separable operation is performed)
- (e5) Multiplicative under tensor product
- (e6) Strong superadditivity.

Examples of entanglement measures

Example 1: Relative entanglement entropy [Lindblad 1972, Uhlmann 1977, Plenio, Vedral 1998,...]:

$$E_R(\omega) = \inf_{\sigma \text{ separable}} H(\omega, \sigma) \ .$$

Here in type I case, $H(\omega, \sigma) = \mathrm{Tr}(\rho_\omega \ln \rho_\omega - \rho_\omega \ln \rho_\sigma)$ = Umegaki's relative entropy. General v. Neumann algebras [Araki 1970s], see below.

Example 2: Distillable entanglement [Rains 2000]:

$$E_D(\omega) = \ln \Big(\text{max. number of Bell-pairs extractable via separable operations from } N \text{ copies of } \omega \Big) \Big/ \text{copy}$$

Example 3: Mutual information [Schrödinger]:

$$E_I(\omega) = H(\omega, \omega_A \otimes \omega_B) \qquad (1)$$

where $\omega_A = \omega \restriction \mathfrak{A}_A$ etc.

Examples of entanglement measures

Example 4: Bell correlations [Bell 1964, Tsirelson 1980, Summers & Werner 1987 ...]

Example 5: Logarithmic dominance [SH & Sanders 2017, Datta 2009]:

$$E_N(\omega) = \ln\Big(\inf\{\|\sigma\| \mid \sigma \geq \omega, \sigma \text{ separable}\}\Big)$$

Example 6: Modular entanglement [SH & Sanders 2017]:

$$E_M(\omega) = \ln\Big(\min(\|\Psi^A\|_1, \|\Psi^B\|_1)\Big) \tag{2}$$

where $\Psi^A : \mathfrak{A}_A \to \mathcal{H}$ given by $a \mapsto \Delta^{1/4} a|\Omega\rangle$, $|\Omega\rangle$ is the GNS-vector representing ω and Δ is the modular operator for the commutant of $\mathfrak{A}_B$ (Here $\|\ .\ \|_1$ is the 1-norm of a linear map.)

Many other examples [Otani & Tanimoto 2017, Christiandl et al. 2004, ...]!

Uniqueness?

For pure states one has basic fact [Donald, Horodecki, Rudolph 2002]:

Uniqueness

For pure states, basically all entanglement measures agree with relative entanglement entropy.

For mixed states, uniqueness is lost. In QFT, we are always in this situation!

Some relationships [SH & Sanders 2017]

Measure	Properties	Relationships	$E(\omega_n^+)$
E_B	OK		$\sqrt{2}$
E_D	OK	$E_D \leq E_R, E_N, E_M, E_I$	$\ln n$
E_R	OK	$E_D \leq E_R \leq E_N, E_M, E_I$	$\ln n$
E_N	OK	$E_D, E_R \leq E_N \leq E_M$	$\ln n$
E_M	mostly OK	$E_D, E_R, E_N \leq E_M$	$\frac{3}{2}\ln n$
E_I	some OK	$E_D, E_R \leq E_I$	$2\ln n$

(Here ω_n^+=Bell state from Example 2)

Modular theory I

Modular theory is a key structural tool in v. Neumann algebra theory. If $\mathfrak{M}$ is a v. Neumann algebra on $\mathcal{H}$ with cyclic and separating vector $|\Omega\rangle$, then one defines S as $(a \in \mathfrak{M})$,

$$S_\omega a|\Omega\rangle = a^*|\Omega\rangle, \quad S_\omega = J\Delta^{1/2} \quad \text{polar decomposition.} \tag{3}$$

Similarly, given two such states, one defines $S_{\omega,\omega'} a|\Omega'\rangle = a^*|\Omega\rangle$, with corresponding polar decomposition ($\rightarrow$ relative modular operator).

Modular (Tomita-Takesaki-) theory

The structural properties of Δ (modular operator) imply many properties of the corresponding entanglement measures such as E_M, E_R, E_I.

Modular theory II

Modular theory

Some structural properties of Δ (modular operator):

1. $\sigma_t(a) = \Delta^{it} a \Delta^{-it}$ leaves $\mathfrak{M}$ invariant. In QFT, if $\mathfrak{M} = \mathfrak{A}(O)$ for certain special O, $\omega =$ vacuum, then σ_t generates the action of spacetime symmetries [Bisognano & Wichmann 1976, Hislop & Longo 1982, Brunetti, Guido & Longo 1993].
2. $\omega \mapsto \|\Delta_\omega^\alpha a\Omega\|^2$ is a concave functional on states for $0 < \alpha < 1/2$ (WYDL concavity).
3. If $\mathfrak{M}_1 \subset \mathfrak{M}_2$ then $\Delta_2^\alpha \leq \Delta_1^\alpha$ (Löwner's theorem)
4. KMS-property: $z \mapsto \omega(a\sigma_z(b))$ can be extended to an analytic function in strip $0 < \Im(z) < 1$ and the boundary values satisfy $\omega(a\sigma_{t+i}(b)) = \omega(\sigma_t(b)a)$.

There are similar properties for the relative modular operator. The relative entropy is related by

$$H(\omega, \omega') = \langle \Omega | \ln \Delta_{\omega,\omega'} \Omega \rangle.$$

Some results

Some results [SH & Sanders 2017]:

1. $d+1$-dimensional CFTs
2. An exact result in $1+1$ CFT [Longo & Xu 2018, Casini & Huerta 2009]
3. Locality of entanglement [SH 2018 (to appear)]
4. Origin of "area law"
5. Exponential decay
6. Charged states
7. $1+1$-dimensional integrable models

CFTs

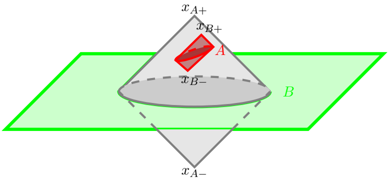


Figure: Nested causal diamonds.

Define conformally invariant cross-ratios u, v by

$$u = \frac{(x_{B+} - x_{B-})^2 (x_{A+} - x_{A-})^2}{(x_{A-} - x_{B-})^2 (x_{A+} - x_{B+})^2} > 0$$

(v similarly) and set

$$\theta = \cosh^{-1}\left(\frac{1}{\sqrt{v}} - \frac{1}{\sqrt{u}}\right) , \quad \tau = \cosh^{-1}\left(\frac{1}{\sqrt{v}} + \frac{1}{\sqrt{u}}\right) .$$

Upper bound

For vacuum state ω_0 in any $3+1$ dimensional CFT with local operators $\{\mathcal{O}\}$ of dimensions $d_{\mathcal{O}}$ and spins $S_{\mathcal{O}}^{L,R}$:

$$E_M(\omega_0) \leq \ln \sum_{\mathcal{O}} e^{-\tau d_{\mathcal{O}}} [2S_{\mathcal{O}}^R + 1]_\theta [2S_{\mathcal{O}}^L + 1]_\theta ,$$

with $[n]_\theta = (e^{n\theta/2} - e^{-n\theta/2})/(e^{\theta/2} - e^{-\theta/2})$.

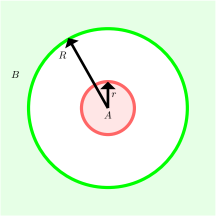


Figure: The regions A and B.

For concentric diamonds with radii $R \gg r$ this gives

$$E_R(\omega_0) \leq E_M(\omega_0) \lesssim N_{\mathcal{O}} \left(\frac{r}{R}\right)^{d_{\mathcal{O}}} ,$$

where $\mathcal{O} =$ operator with the smallest dimension $d_{\mathcal{O}}$ and $N_{\mathcal{O}} =$ its multiplicity.

Tools: Hislop-Longo theorem, Tomita-Takesaki theory

Exact result in 1+1

An exact result was recently obtained by [Longo & Xu 2018] building on previous ideas of [Casini & Huerta 2009, Calabrese, Cardy, Tonni 2009/11]. They prove rigorously that for a free Dirac field on a lightray (or related theories via canonical constructions in CFT):

Free fermions

For $A, B =$ union of disjoint intervals, $\mathrm{dist}(A, B) > 0$, one has

$$E_I(\omega_0) = -\frac{c}{3} \ln u$$

where u is the analogue of the conformally invariant cross ratio (on light ray), and where ω_0 is vacuum (and $c = 1/2$ for free fermion).

As a consequence, $E_R(\omega_0) \leq -\frac{c}{3} \ln u$.

Ingredients of proof: CAR, Kosaki-formula, ...

Locality of entanglement I

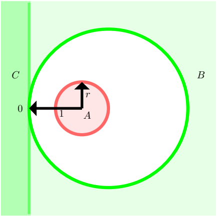


Figure: The regions A, B, C.

Consider regions B, C touching at the origin of Minkowski. $A \subset B'$ is a diamond of radius $r < 1$ whose center is at distance $= 1$ away from origin. $\lambda A =$ scaled diamond.

Assume: QFT has scaling limit which is a CFT.

Theorem [SH in preparation]

If k is the largest eigenvalue of the extrinsic curvature tensor of ∂B where B and C touch, then as $\lambda \to 0$,

$$\left| E_M(\omega_{\lambda A \otimes C}) - E_M(\omega_{\lambda A \otimes B}) \right| \leq \text{cst.}\, (k\lambda)^{\frac{1}{2}}\, Z_{\text{CFT}}(\tau = \cosh^{-1} r^{-1})$$

for some explicit constant.

Locality of entanglement I

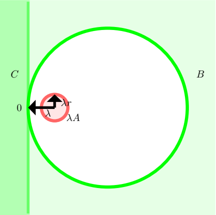


Figure: The regions $\lambda A, B, C$.

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for some explicit constant.

Locality of entanglement I

Remark: I conjecture that upper bound is optimal. If B and C touch at a point of a bifurcate Killing horizon, then upper bound is same with only change $(k\lambda)^{\frac{\kappa}{2}}$ with κ the surface gravity of bh.

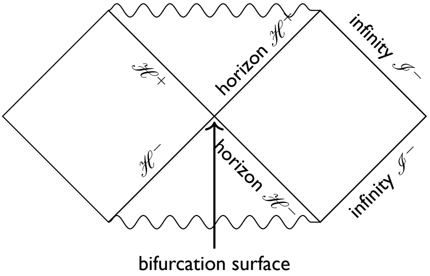

Figure: Spacetime with bifurcate Killing horizon.

Locality of entanglement II

How might one prove such a theorem? At the heart of the proof is the following general result (for a related result see [Fredenhagen 1985]):

Key lemma [SH in preparation]

Let $\mathfrak{M}_1 \subset \mathfrak{M}_2$ with common cyclic and separating $|\Omega\rangle$. Assume $\sigma_{2,t}(a) \in \mathfrak{M}_1$ for $|t| \le \tau$ for some $a \in \mathfrak{M}_1$. Then for $0 < \alpha < 1/2$,

$$0 \le \|\Delta_1^\alpha a\Omega\|^2 - \|\Delta_2^\alpha a\Omega\|^2 \le \mathrm{cst.}\ (1 + \pi\tau) e^{-\pi\tau} \|\Delta_2^\alpha a\Omega\|^2 \tag{4}$$

for some explicit constant dep. on α.

The proof of the theorem is obtained by combining this lemma with:

- The Bisognano-Wichmann theorem, choosing C to be a half-plane and $\mathfrak{M}_1 = \mathfrak{A}'_C, \mathfrak{M}_2 = \mathfrak{A}'_B$. Then $\tau \sim |\ln(k\lambda)|/2\pi$ can be estimated for $a \in \mathfrak{A}_{\lambda A}$ since modular flow of C has geometric nature.
- Basic properties of the nuclear 1-norm.
- Previous estimates of E_M in CFTs.

Free massive QFTs

A and B regions in a static time slice in ultra-static spacetime, $\mathrm{d}s^2 = -\mathrm{d}t^2 + h(\text{space})$; lowest energy state: ω_0. Geodesic distance: r

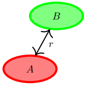

Figure: The the systems A, B

Upper bounds (decay + area law)

Dirac field: As $r \to 0$

$$E_R(\omega_0) \lesssim \mathrm{cst.} |\ln(mr)| \sum_{j \geq d-1} r^{-j} \int_{\partial A} a_j$$

where a_j curvature invariants of ∂A. Lowest order $\Longrightarrow$ area law.
Klein-Gordon field: As $r \to \infty$ decay

$$E_R(\omega_0) \lesssim \mathrm{cst.} e^{-mr/2}$$

(Dirac: [Islam, SH & Sanders])

Proof of exponential decay:

1. First show that

$$E_R(\omega_0) \leq -4 \sum_{\pm} \operatorname{Tr} \ln(1 - |(1 - Q_{B'\mp})Q_{A\pm}|^{\frac{1}{2}})$$

   for certain projection operators onto subspaces of $L^2(\mathcal{C})$ associated w/ A, B' ($\rightarrow$ modular theory).

2. Then show that the estimation boils down to that of operator norms

$$\|C^\alpha \chi_A C^\beta (1 - \chi_B)\|$$

   where $C = (-\nabla^2_{\mathcal{C}} + m^2)^{-1}$, where $\alpha, \beta \in \mathbb{R}$ (depending on the dimension). χ_A is a smoothed out indicator function of A, similarly B.

3. Use "finite propagation speed" [Fefferman et al. 1986] property of $\exp it(-\nabla^2_{\mathcal{C}} + m^2)^{1/2}$ and Fourier representation $(X^2 + \lambda^2)^\alpha = \int dt f(t) e^{itX}$. Integration range for t effectively cut off to $|t| > r$. $\Rightarrow$ exponential decay in r.

We expect our methods to yield similar results to hold generally on spacetimes with bifurcate Killing horizon:

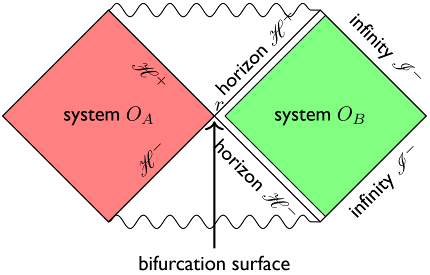

Figure: Spacetime with bifurcate Killing horizon.

Charged states

A and B regions, ω any normal state in a QFT in $d+1$ dim. $\chi^*\omega$ state obtained by adding "charges" χ in A or B.

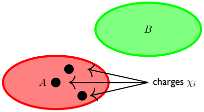

Figure: Adding charges to state in A

Upper bound

$$0 \leq E_R(\omega) - E_R(\chi^*\omega) \leq \ln \prod_i \dim(\chi_i)^{2n_i} \ ,$$

n_i: $\#$ irreducible charges χ_i type i, and

$$\dim(\chi_i) = \text{quantum dimension} = \sqrt{\text{Jones index}}$$

Remark: Same inequality for E_M.

Index-statistics theorem [Longo 1989/90], Jones subfactor theory, Pimsner-Popa-inequality, Doplicher-Haag-Roberts theory; Naaijkens talk

Examples

Example: $d = 1$, Minimal model type $(p, p+1)$, χ irreducible charge of type (n, m)

$$0 \leq E_R(\omega) - E_R(\chi^*\omega) \leq \ln \frac{\sin\left(\frac{\pi(p+1)m}{p}\right) \sin\left(\frac{\pi p n}{p+1}\right)}{\sin\left(\frac{\pi(p+1)}{p}\right) \sin\left(\frac{\pi p}{p+1}\right)} \ .$$

Example: $d > 1$, general QFT, irreducible charge χ with Young tableaux statistics $\begin{array}{|c|c|c|c|c|c|} \hline 8 & 6 & 5 & 4 & 2 & 1 \\ \hline 5 & 3 & 2 & 1 & & \\ \cline{1-4} 1 & & & & & \\ \cline{1-1} \end{array}$.

$$0 \leq E_R(\omega) - E_R(\chi^*\omega) \leq 2 \ln 5,945,940$$

Area law in asymptotically free QFTs

A and B regions separated by a thin corridor of diameter $\varepsilon > 0$ in $d+1$ dimensional Minkowski spacetime, vacuum $\omega_0 =$ vacuum.

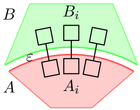

Figure: The the systems A, B

Result (“area law”)

Asymptotically, as $\varepsilon \to 0$

$$E_R(\omega_0) \gtrsim \begin{cases} D_2 \cdot |\partial A| / \varepsilon^{d-1} & d > 1, \\ D_2 \cdot \ln \frac{\min(|A|, |B|)}{\varepsilon} & d = 1, \end{cases}$$

where $D_2 =$ distillable entropy E_D of an elementary “Cbit” pair

Tools: Strong super additivity of E_D, bounds [Donald, Horodecki, Rudolph 2002], also [Verch, Werner 2005, Wolf, Werner 2001/06,HHorodecki 1999]

Integrable models

These models (i.e. their algebras $\mathfrak{A}_A$) are constructed using an "inverse scattering" method from their 2-body S-matrix, e.g.

$$S_2(\theta) = \prod_{k=1}^{2N+1} \frac{\sinh\theta - i\sin b_k}{\sinh\theta + i\sin b_k} \,,$$

by [Schroer & Wiesbrock 2000, Buchholz & Lechner 2004, Lechner 2008, Allazawi & Lechner 2016, Cadamuro & Tanimoto 2016].
$b_i =$ parameters specifying model, e.g. sinh-Gordon model ($N = 0$).

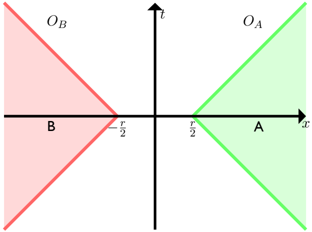

Figure: The regions A, B.

Upper bound

For vacuum state ω_0 and mass $m > 0$:

$$E_R(\omega_0) \leq E_M(\omega_0) \lesssim \mathrm{cst}.e^{-mr(1-k)} \; .$$

for $mr \gg 1$. The constant depends on the scattering matrix S_2, and $k > 0$.

Idea of the proof: E_M is related to the log of the 1-norm of the linear map

$$\mathfrak{A}_A \ni a \mapsto \Delta^{1/4} a|\Omega\rangle \in \mathcal{H},$$

where Δ is the modular operator of B'. The corresponding modular flow acts geometrically by Bisognano-Wichmann. In fact, the norm can be estimated explicitly using an explicit construction of the operator algebras $\mathfrak{A}_A, \mathfrak{A}_B$ on the S_2-symmetric Fock space $\mathcal{H}$, relying on techniques of [Lechner 2008, Allazawi & Lechner 2016]

In this talk, I have

- Explained what entanglement is, and how it can be used.
- Explained what an entanglement measure is, and given concrete examples
- Explained how entanglement arises in Quantum Field Theory, and why there always has to be a finite safety corridor between the systems.
- Evaluated (in the sense of upper and lower bounds) a particularly natural entanglement measure in several geometrical setups, quantum field theories and states of interest.
- Given some idea how modular theory (Tomita-Takesaki theory) comes in.

Worth further study: relation with the considerable literature on v. Neumann entropy in the theoretical physics literature! Especially:

- 2d CFTs Calabrese, Cardy, Nozaki, Numasawa,Takayanagi,...
- 2d integrable models Calabrese, Cardy, Doyon, ...
- Modular theory, c-theorems: Casini, Huerta,...
- Holographic methods Hubeny, Myers, Rangamani, Ryu, Takayanagi,...