Operator Algebras and Construction of Quantum Field Theories

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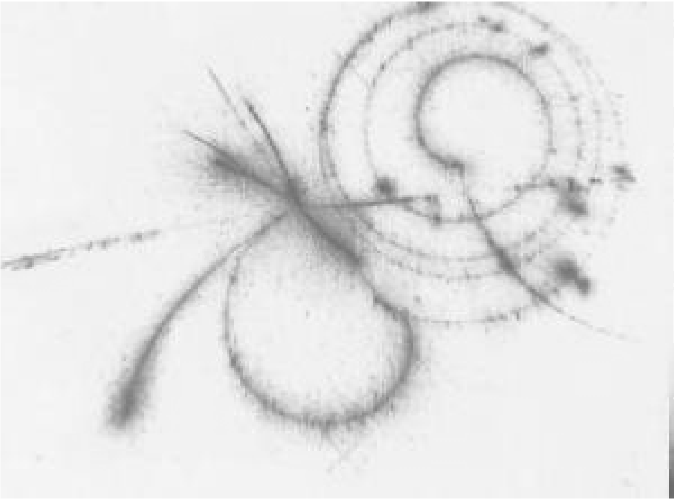


Outline

- Mathematical approaches to QFT
- Algebraic formulation of QFT
- Monads of algebraic QFT
- Deformations of dynamical systems
- Construction of QFTs by deformation
- Conclusions

Mathematical approaches to QFT

There are different strategies: Depending on ones goals and taste one proceeds from

- a classical field theory on spacetime $\mathbb{R}^d$ (Lagrangean) and a method of quantizing it
- a vacuum functional on a (non–commutative) tensor algebra of test functions on $\mathbb{R}^d$
- a suitable measure on a space of distributions on Euclidean space $\mathbb{R}^d_E$ (followed by analytic continuation to $\mathbb{R}^d$)
- a Poincaré covariant and causal net of operator algebras on $\mathbb{R}^d$

These approaches describe the same physics. But they shed different light on the interpretation and construction of QFTs.

Algebraic formulation of QFT

Microscopic systems require QT description:

States (ensembles): normalized vectors $\Phi \in \mathcal{H}$ (Hilbert space)

Observables (instruments): selfadjoint operators $A \in \mathcal{B}(\mathcal{H})$ (Algebra)

Theoretical Predictions:

Expectations (mean values of raw data): $\langle \Phi, A\,\Phi \rangle \in \mathbb{R}$.

Measurements take place in subregions $\mathcal{R}$ of spacetime.

This refined information is provided by any QFT by specifying a map

$$\mathbb{R}^d \supset \mathcal{R} \longmapsto \mathcal{A}(\mathcal{R}) \subset \mathcal{B}(\mathcal{H})$$

subject to the condition of isotony (net)

$$\mathcal{A}(\mathcal{R}_1) \subset \mathcal{A}(\mathcal{R}_2) \quad \text{if} \quad \mathcal{R}_1 \subset \mathcal{R}_2\,.$$

Measurements conform with symmetries and causal structure of spacetime.

Isometries of $(\mathbb{R}^d, g)$: Poincaré group $\mathcal{P}_+^\uparrow = \mathbb{R}^d \rtimes SO_o(1, d-1)$.

- Covariance:
  Automorphic action of $\mathcal{P}_+^\uparrow$ on observables is induced by unitary representation U of $\mathcal{P}_+^\uparrow$ s.t.

$$U(\lambda)\mathcal{A}(\mathcal{R})U(\lambda)^{-1} = \mathcal{A}(\lambda\mathcal{R}), \quad \lambda \in \mathcal{P}_+^\uparrow$$

- Stability: (elementary states)
  sp $U \restriction \mathbb{R}^1 \subset \mathbb{R}_+$ and there is a U–invariant state $\Omega \in \mathcal{H}$ (vacuum)
- Locality (Einstein Causality):

$$\mathcal{A}(\mathcal{R}_1) \subset \mathcal{A}(\mathcal{R}_2)' \quad \text{if } \mathcal{R}_1 \subset \mathcal{R}_2' .$$

$\mathcal{A}'$ commutant of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$, $\mathcal{R}'$ causal complement of $\mathcal{R}$ in $(\mathbb{R}^d, g)$.

Fundamental insight: **given** $(\{\mathcal{A}(\mathcal{R})\}_{\mathcal{R}\subset\mathbb{R}^d}, U, \Omega)$ **one can extract**

- particle content
- particle statistics, symmetries
- scattering data
- underlying quantum fields
- short distance properties (quarks)
- properties of thermal states ...

Algebraic approach provides a convenient framework for structural analysis. But it may seem inappropriate as a constructive tool ...

Monads of algebraic QFT

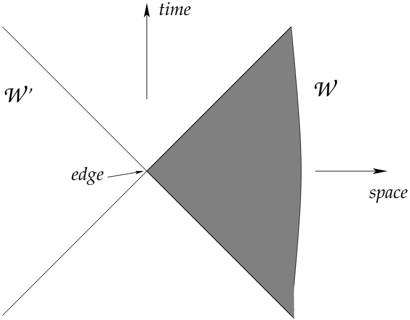
Ingredients required in AQFT:

- suitable unitary representation U of $\mathcal{P}_+^\uparrow$
- system of algebras $\{\mathcal{A}(\mathcal{R})\}_{\mathcal{R}\subset\mathbb{R}^d}$ compatible with action of U

Strategy for their construction:

- Specify stable particle content of theory and construct corresponding representation U (*e.g.* on Fock space $\mathcal{H}$)
- Identify a **single** von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ subject to certain compatibility conditions such that it can be interpreted as algebra of observables in a particular wedge shaped region $\mathcal{W} \subset \mathbb{R}^d$ (Monad).
- Generate the system of algebras $\{\mathcal{A}(\mathcal{R})\}_{\mathcal{R}\subset\mathbb{R}^d}$ from these data.

Wedge regions: Standard wedge $\mathcal{W} = \{x \in \mathbb{R}^d : x_1 \geq |x_0|\}$



Remarks:

Poincarè group $\mathcal{P}_+^\uparrow$ acts transitively on the set of wedges in $\mathbb{R}^d$, $d > 2$
There exist $\mathcal{S}, \mathcal{S}' \subset \mathcal{P}_+^\uparrow$ such that

$$\lambda\mathcal{W} \subset \mathcal{W}\,,\ \lambda \in \mathcal{S} \ \text{ and } \ \lambda'\mathcal{W} \subset \mathcal{W}'\,,\ \lambda' \in \mathcal{S}'\,.$$

Compatibility conditions on $(\mathcal{M}, U, \Omega)$:

- $U(\lambda)\mathcal{M}U(\lambda)^{-1} \subset \mathcal{M}$ whenever $\lambda\mathcal{W} \subset \mathcal{W}$,
- $U(\lambda')\mathcal{M}U(\lambda')^{-1} \subset \mathcal{M}'$ whenever $\lambda'\mathcal{W} \subset \mathcal{W}'$
- $\mathcal{M}\Omega$ dense in $\mathcal{H}$.

Terminology: $(\mathcal{M}, U, \Omega)$ is a causal triple

Resulting observable algebras:

$$\mathcal{A}(\lambda\mathcal{W}) \doteq U(\lambda)\mathcal{M}U(\lambda)^{-1}\,, \quad \lambda \in \mathcal{P}_+^\uparrow\,.$$

Note: definition consistent.

$$\mathcal{R} = \bigcap_{\mathcal{W}.\supset\mathcal{R}} \mathcal{W}. \longmapsto \mathcal{A}(\mathcal{R}) \doteq \bigcap_{\mathcal{W}.\supset\mathcal{R}} \mathcal{A}(\mathcal{W}.)\,.$$

Proposition

Given $(\mathcal{M}, U, \Omega)$, this map defines a local and covariant net on $(\mathbb{R}^d, g)$.

Problem of constructive AQFT:

Given (U, Ω), exhibit algebras $\mathcal{M}$ such that $(\mathcal{M}, U, \Omega)$ is a causal triple.

Remarks:

- All QFTs with given particle content arise in this way.
- Internal algebraic structure of admissible algebras $\mathcal{M}$ is known to be universal (hyperfinite III_1 factor).

Algebraic constructions of causal triples:

- Free QFT; new examples [Brunetti, Guido, Longo]
- Infinity of integrable models in $d = 2$; proof of complete particle interpretation [Schroer; Lechner]
- First examples of non–free QFTs for any d [Grosse, Lechner]
- Deformation of QFTs [Buchholz, Lechner, Summers; Dybalski, Tanimoto]

Deformations of dynamical systems

Input: C*–dynamical systems

- $\mathcal{C}$ unital C*–algebra,
- $\alpha : \mathbb{R}^d \to \mathrm{Aut}\,\mathcal{C}$ acting continuously.

Here: $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$ algebra of all operators transforming continuously under the action $\alpha_x(C) \doteq U(x)CU(x)^{-1}$, $x \in \mathbb{R}^d$.

Goal: Deformation of $(\mathcal{C}, \alpha)$ without changing α.

1. Rieffel deformation:
$\mathcal{C}^\infty \subset \mathcal{C}$ smooth subalgebra. Pick skew symmetric matrix Q on $\mathbb{R}^d$ and define new product $\times_Q$ on $\mathcal{C}^\infty$

$$C_1 \times_Q C_2 \doteq (2\pi)^{-d} \iint dxdy\, e^{-ixy}\, \alpha_{Qx}(C_1)\, \alpha_y(C_2)\,.$$

Elementary observation:
For given polynomial P there exists some polynomial P' (and *vice versa*) such that

$$P(x, y)\, e^{-ixy} = P'(\partial_x, \partial_y)\, e^{-ixy} .$$

Thus for smooth and bounded functions $x, y \longmapsto F(x, y)$ with values in some Banach space there exist the strong limits

$$\lim_{\varepsilon \searrow 0} \iint dxdy\, e^{-ixy} e^{-\varepsilon(x^2+y^2)}\, F(x, y) .$$

Results:

- $\times_Q$ defines an associative product on $\mathcal{C}^\infty$
- $\times_Q$ compatible with original *–operation
- $(\mathcal{C}^\infty, \times_Q)$ admits a C*–norm $\|\cdot\|_Q$; completion $(\mathcal{C}_Q, \times_Q)$
- α extends continuously from $\mathcal{C}^\infty$ to $(\mathcal{C}_Q, \times_Q)$

Thus $((\mathcal{C}_Q, \times_Q), \alpha)$ is a deformation of the dynamical system $(\mathcal{C}, \alpha)$.

Remarks:

- Method used for quantization of classical systems
- deformations for actions of non–abelian groups are under investigation

2. Warped convolution:

Input: $(\mathcal{C}^\infty, U)$ and skew symmetric matrix Q

$$ {}_QC \doteq (2\pi)^{-d} \iint dxdy\, e^{-ixy}\, \alpha_{Qx}(C)\, U(y) = \int \alpha_{Qp}(C)\, dE(p) \ , $$
$$ C_Q \doteq (2\pi)^{-d} \iint dxdy\, e^{-ixy}\, U(x)\, \alpha_{Qy}(C) = \int dE(p)\, \alpha_{Qp}(C) $$

Meaningful definition of left/right warped convolutions.

Terminology: Convolution of C and dE, warped by the action of Q

Results:

- Left and right warped convolutions are equal: $C_Q = {}_Q C$.
- Warped convolution is symmetric: $(C_Q)^* = (C^*)_Q$.
- Warped convolution provides a representation π_Q of $(\mathcal{C}^\infty, \times_Q)$:

$$\pi_Q(C) \doteq C_Q,\ C \in \mathcal{C}^\infty$$

In particular

$$\pi_Q(C_1)\,\pi_Q(C_2) = \pi_Q(C_1 \times_Q C_2)\,.$$

Theorem

π_Q extends to a faithful representation of $((\mathcal{C}_Q, \times_Q), \alpha)$ on $\mathcal{H}$.

In particular $\|C_Q\| < \infty$.

Gain? Warped convolution more convenient in QFT, for:

- still meaningful if algebra is **not** stable under the action of α (think of $\mathcal{M}$)
- products $C_{1Q_1} C_{2Q_2}$ meaningful for $Q_1 \neq Q_2$
- allows study of relations resulting from properties of sp U

Properties involving different Qs:

- **Covariance:** V unitary and $VU(x)V^{-1} = U(Mx)$. Then

$$VC_Q V^{-1} = (VCV^{-1})_{MQM^T}\,, \quad C \in \mathcal{C}^\infty\,.$$

- **Spectral commutativity**:

  Let $C, C' \in \mathcal{C}^\infty$ be such that for all $p, q \in \mathrm{sp}\,\boldsymbol{U}$

$$\alpha_{Qp}(C)\alpha_{-Qq}(C') = \alpha_{-Qq}(C')\alpha_{Qp}(C)\,.$$

  Then

$$C_Q\, C'_{-Q} = C'_{-Q}\, C_Q\,.$$

- **Group structure:**

$$(C_{Q_1})_{Q_2} = C_{Q_1+Q_2}\,.$$

Construction of QFTs by deformation

Application of warping procedure to causal triples $(\mathcal{M}.U,\Omega)$:

$$\mathcal{M}_Q \doteq \{A_Q : A \in \mathcal{M} \cap \mathcal{C}^\infty\}''$$

Problem: Adjust Q such that $(\mathcal{M}_Q.U,\Omega)$ is again a causal triple.

Admissible semigroup:

$$Q \doteq \begin{pmatrix} 0 & \kappa & \mathbf{0} \\ -\kappa & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \qquad \kappa > 0\,.$$

Observations:

- If $\lambda = (x,\Lambda) \in \mathcal{P}_+^\uparrow$ s.t. $\lambda\mathcal{W} \subset \mathcal{W}$, then $\Lambda Q \Lambda^T = Q$.
- If $\lambda = (x,\Lambda) \in \mathcal{P}_+^\uparrow$ s.t. $\lambda\mathcal{W} \subset \mathcal{W}'$, then $\Lambda Q \Lambda^T = -Q$.
- $Q\,\mathrm{sp}U \subset W$ and $-Q\,\mathrm{sp}U \subset W'$.

Theorem

Let $(\mathcal{M}, U, \Omega)$ *be a causal triple and let* Q *be admissible. Then* $(\mathcal{M}_Q, U, \Omega)$ *is also a causal triple.*

- Starting from any QFT (e.g. a free theory) one arrives by warped convolution at deformed theories for any $d \geq 2$.
- Resulting theories are non–isomorphic to each other in general.
- Particle content of the theory does not change, but scattering data (S–matrix) changes.
- Algebras $\mathcal{A}(\mathcal{R})$ for bounded regions $\mathcal{R}$ are small.
  (Interpretation as theories living on non–commutative Minkowski space.)

Conclusions

Theory of operator algebras sheds new light on constructive problems in QFT

- Task: Exhibit for chosen representation U algebra(s) $\mathcal{M}$ such that $(\mathcal{M}, U, \Omega)$ is a causal triple.
- Warped convolution provides further examples of such triples.
  Work on other kinds of deformations in progress.
- Method can be transfered to more general (curved) spacetimes

Theory of operator algebras provides mathematical tools which are complementary to those used in other approaches to QFT. Combining them will hopefully lead to the mathematical consolidation of QFT.

May the CMTP florish and contribute to this important goal!