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LOCAL DYNAMICS OF HOLOMORPHIC DIFFEOMORPHISMS

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ABSTRACT. This is a survey about local holomorphic dynamics, from Poincaré's times to nowadays. Some new ideas on how to relate discrete dynamics to continuous dynamics are also introduced. It is the text of the talk given by the author at the XVII UMI Congress at Milano.

SUNTO. Questo é un sunto dello stato dell'arte della dinamica olomorfa complessa dai tempi di Poincaré ai giorni nostri. Sono inoltre indicate alcune nuove idee per mettere in relazione la dinamica discreta con la dinamica continua. É il testo della conferenza tenuta dall'autore nel XVII Convegno dell'UMI a Milano.

PROLOGUE

Let M be a complex manifold and $f : M \to M$ a holomorphic map. The study of the behavior of the sequence of iterates of f, $\{f^{\circ k}\}$, is what is nowadays called *holomorphic (discrete) dynamics*. This subject has been studied since the time of Schöder for local dynamics and Fatou and Julia in case of rational mappings of the complex projective line. Much of this theory has been used and improved later by people interested in the *continuous dynamics* of holomorphic foliations, relating dynamics of vector fields in $\mathbb{C}^2$ with the dynamics of holomorphic mappings by means of the Poincaré time one map.

The study of holomorphic dynamics can be done both form the global and the local point of view. From the global point of view one is interested in finding invariant sets for the map and studying their properties. A simple type of (forward) invariant set is given by a fixed point of the map. The forward orbit of such a point is the point itself, but the backward orbit might be very complicated. Trying to simplified the situation one can consider only the behavior of points nearby the fixed point. This type of study is known as local dynamics.

Local dynamics thus uses a magnifying glass to understand what is going on near the fixed point. Therefore, instead of considering maps of a manifold we can just study germs of diffeomorphisms at the fixed point (the ambient will usually be $\mathbb{C}^n$, but one can also study singular ambient spaces). This has the value that, contrarily to the global situation, one can often explicitly write down examples on which figure out the theory.

The best situation one can hope to have is *linearization* of the germ. This means that suitably changing coordinates the map becomes a linear transformation. If the change of coordinates used to linearize the germ is holomorphic than the linear transformation obtained is the differential of the germ at the fixed point (up to conjugation). However if the change of coordinates involved is only continuous then the linear transformation might not be the differential. Holomorphic linearization is the dream of people that study local holomorphic dynamics, for one can really think of the map as a linear transformation. Even topological linearization is useful (for instance it provides trajectories and behavior of orbits) and sometimes it may be useful also to have just formal linearization. Anyhow, the differential is the map which first approximates the dynamics of the map, and thus it is natural to classify and study dynamics according to the spectrum of the differential itself.

As we will see, a generic germ of holomorphic diffeomorphism is holomorphically linearizable. Unfortunately, the non-generic situation comes out often in celestial mechanics and physical problems. Thus one is forced to understand non-linearizable dynamical systems. These are not completely understood, even if from the pioneeristic work of Fatou, Dulac and Poincaré much has been done.

The aim of these notes is to provide a survey on the state of art about local holomorphic dynamics, trying to face on the several ideas appeared on the subject.

The notes are based on the talk I gave at the XVII Congress of UMI in Milano. I wish to thank the organizers for having invited me and for the opportunity of writing these notes.

1. LOCAL DYNAMICS IN DIMENSION ONE

Let f be a germ of holomorphic diffeomorphism at the origin of $\mathbb{C}$ fixing 0. Thus we can expand f as $f(\zeta) = \lambda\zeta + \dots$ where $\lambda \in \mathbb{C} \setminus \{0\}$.

As one can expect the number λ discriminates the local dynamics.

1.1. **Hyperbolic case:** $|\lambda| \neq 0, 1$**.** The main result is due to Königs in 1884 (see, *e.g.* [22]) who solved the so-called Schröder equation

$$\sigma \circ f = \lambda\sigma, \tag{1.1}$$

in case $|\lambda| < 1$ (if $|\lambda| > 1$ one can solve a similar functional equation for f^{-1}). This means that there exists a unique holomorphic diffeomorphism σ such that $\sigma(0) = 0, \sigma'(0) = 1$ which conjugates f to the function $\zeta \mapsto \lambda\zeta$. Therefore the dynamics of f can be read in this new coordinates, and one sees that for any point ζ_0 near to 0 then $f^{\circ k}(\zeta_0) \to 0$ as $k \to \infty$ following a spiralizing or a linear path according to whether λ is complex or pure real.

It should be mention another interpretation of (1.1). Suppose that f is holomorphic on all the unit disc $\Delta = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. Let $H^p(\delta)$ be the p-Hardy space on Δ. One can define a linear operator $C_f : H^p(\Delta) \to H^p(\Delta)$ as $C_f(h) = h \circ f$ (see, *e.g.*, [48]). By the Littlewood's subordination lemma one can show that C_f is actually continuous. Such an operator is called a *composition operator*. Then (1.1) is equivalent to $C_f(\sigma) = \lambda\sigma$. Namely σ is an eigenvalue of C_f. The dynamics of f is strictly related to the functional analysis properties of C_f. We invite the interested reader to read [48] for more on this subject.

¿From the point of view of holomorphic dynamics, having a *holomorphic linearization*, as a solution of (1.1), is the best one can hope. In particular it is not difficult to see that if $\lambda_1 \neq \lambda_2$ then $f_1(\zeta) = \lambda_1\zeta + \dots$ and $f_2(\zeta) = \lambda_2\zeta + \dots$ are not holomorphic conjugated each other, and thus their dynamics is different from a holomorphic point of view. In particular the space of holomorphic parameters for hyperbolic germs is $\mathbb{C} \setminus [\{0\} \cup \partial\Delta]$. From the point of view of topology however the situation is different: one can always find a topological conjugation between any two hyperbolic germs with both derivatives at 0 of modulo less than 1 (respectively, both with modulo greater than 1). Therefore the space of topological parameters is reduced to only two points.

1.2. **Parabolic case: $|\lambda| = 1$, $\lambda^k = 1$ for some $k \in \mathbb{N}$.** This case can be considered as the "resonant case", as it will be clear later. Indeed, one first tries to linearize the germ using *formal series*, and then hope to make them converging. However, the fact that $\lambda^{k+1} = \lambda$ prevents the possibility to kill (even formally) all the terms. Indeed it is not difficult to show that

Proposition 1.1. *The map f is holomorphically conjugated to $\zeta \mapsto \lambda\zeta$ if and only if $f^n(\zeta) = \zeta$ for some $n \in \mathbb{N}$.*

Thus, linearizable parabolic germs are not many. However the dynamics can be still well understood, thanks to the work of Leau and Fatou (see, *e.g.*, [22]). First we remark that $f^k(\zeta) = \zeta + O(\zeta^2)$. Thus essentially one can recover the case $\lambda \neq 1$ from the case $\lambda = 1$. In this case the Leau-Fatou theorem states that it is possible to find invariant simple connected domains containing 0 on the boundary such that on each domain the map is conjugated to a parabolic automorphism of the domain itself and each point of such a domain is attracted to 0. These domains are called *petals* and their existence is predicted by the *Leau-Fatou Flower Theorem*. To give a simple statement of such a result, we note that if $f(\zeta) = \zeta + a_r\zeta^r + O(\zeta^{r+1})$ with $r > 1$ and $a_r \neq 0$, it is possible to perform a holomorphic change of variables in such a way that f becomes conjugated to $\zeta \mapsto \zeta + \zeta^r + O(\zeta^{r+1})$. The number r is the *order* of f at 0. With these preliminary considerations at hand we have

Theorem 1.2 (Leau-Fatou Flower Theorem)**.** *Let $f(\zeta) = \zeta + \zeta^r + O(\zeta^{r+1})$ with $r > 1$. Then there exist $(r-1)$ domains called petals, P_j, symmetric with respect to the $r-1$ directions $\arg\zeta = 2\pi q/(r-1)$, $q = 0, \ldots, r-2$ such that $P_j \cap P_k = \emptyset$ for $j \neq k$, $0 \in \partial P_j$, each P_j is biholomorphic to the right-half plane H, and for all $\zeta \in P_j$ it follows $f^{\circ k}(\zeta) \to 0$ as $k \to \infty$. Moreover for all j, the map $f|_{P_j}$ is holomorphically conjugated to the parabolic automorphism $\zeta \mapsto \zeta + i$ on H.*

Now, $f^{-1}(\zeta) = \zeta - \zeta^r + O(\zeta^{r+1})$. Thus, applying Theorem 1.2 to f^{-1} one gets $r-1$ attracting petals Q_j for f^{-1} symmetric with respect to the $r-1$ directions $\arg\zeta = (2q+1)\pi q/(r-1)$, $q = 0, \ldots, r-2$. Notice that these directions are exactly the bisectrices of the angles between two consecutive attracting directions for f. It is clear that the Q_j's are repelling petals for f, intersecting the P_j's and $\bigcup_j P_j \cup Q_j \cup \{0\}$ is an open neighborhood of 0 in $\mathbb{C}$. Therefore now the dynamics of f can be read easily.

If $\lambda \neq 1$ (and $\lambda^k = 1$) then f acts as a permutation on the petals of f^k, which are thus a multiple of k. It should be notice however that if $f(\zeta) = \lambda\zeta + a_r\zeta^r + \ldots$ with $a_r \neq 0$, then the number of petals might be different from r. Indeed it may happen that f^k has order $> r$ at 0.

We saw that there is no hope to obtain a holomorphic linearization for parabolic germs. However one may ask what happens from the topological point of view, and more generally which are the classes of holomorphic conjugacy. Both questions have been answered. The topological classification is in fact pretty simple (even if not easy to obtain) , and it is due to C. Camacho [18] and, independently, by Shcherbakov [49].

Theorem 1.3 (Camacho, Shcherbakov)**.** *Let* $f(\zeta) = \lambda\zeta + O(\zeta^2)$ *be holomorphic,* $\lambda^n = 1$ *for some* $n \in \mathbb{N}$ *and, if* $n > 1$ *assume* $\lambda^m \neq 1$ *for* $1 \leq m < n$*. Then*

(i) *either* $f^n(\zeta) = \zeta$*,*
(ii) *or there exists* $k \in \mathbb{N}$ *such that* f *is topological conjugate to* $\zeta \mapsto \lambda\zeta(1 + \zeta^{nk})$*.*

Remark 1.4. If $f(\zeta) = \zeta + a_r\zeta^r + O(\zeta^{r+1})$ with $a_r \neq 0$ then f is topological conjugate to $\zeta \mapsto \zeta + \zeta^r$.

The proof of the theorem shows actually that one can topologically conjugate f to an automorphism of a suitable Riemann surface. Camacho's original proof is itself very beautiful and provides some more hints on the dynamics of the map. Therefore we provide some details of the proof, at least for the case $\lambda = 1$.

Sketch of the proof of Theorem 1.3 for $\lambda = 1$. Let $f(\zeta) \neq \zeta$ be given by $f(\zeta) = \zeta + \zeta^{m+1} + O(\zeta^{m+2})$. By Theorem 1.2 the union of petals $\bigcup_j P_j \cup Q_j$ is an open set around 0, and on each petal the germ f is conjugated to an automorphism of such a petal.

The idea is now to consider each petal as a chart of a suitable Riemann surface in such a way that the conjugations on each chart glue together to give a global conjugation of f to an automorphism of the Riemann surface. More precisely, let $\mathcal{S}_m$ be the Riemann surface of the function $\zeta \mapsto \zeta^{-m}$. The surface $\mathcal{S}_m$ can be defined as $\mathcal{S}_m = \{(z,w) \in \mathbb{C}^* \times \mathbb{C}^* : w = z^{-m}\}$. Let $\mathbb{C}^*_r = \{\zeta \in \mathbb{C}^* : |\zeta| < r\}$ for a small $r > 0$. Let $\mathcal{S}^r_m = \pi_1^{-1}(\mathbb{C}^*_r)$, where $\pi_1(z,w) = z$. Then we can well define a holomorphic injective map $F : \mathcal{S}^r_m \to \mathcal{S}_m$ as $F = \pi_1^{-1} \circ f \circ \pi_1$. Now notice that $\pi_2 : \mathcal{S}_m \to \mathbb{C}^*$, where $\pi_2(z,w) = w$, is a m-th covering. In particular π_2 is a biholomorphism on $\pi_1^{-1}(P_j) \cap \mathcal{S}_m$ (and $\pi_1^{-1}(Q_j) \cap \mathcal{S}_m$), whose inverse, which with some abuse of notation we denote by $\pi_2^{-1}|_{\pi_1^{-1}(P_j)}$, is given by (the appropriate branch of) $z \mapsto z^{-1/m}$. If we use $(\pi_2|_{\pi_1^{-1}(P_j)}, \pi_1^{-1}(P_j) \cap \mathcal{S}_m)$ as a local chart on $\mathcal{S}_m$, and take into account that by Theorem 1.2 the domain P_j is f-invariant, we get

$$\pi_2|_{\pi_1^{-1}(P_j)} \circ F \circ \pi_2^{-1}|_{\pi_1^{-1}(P_j)}(z) = \pi_2 \circ f(z^{-1/m}) = [f(z^{-1/m})]^{-m} = z - m + cz^{-1/m} + \ldots,$$

where the branch of $z^{-1/m}$ is chosen so that $i^{-1/m} \in P_j$. We define an injective holomorphic map $G : \mathcal{S}^r_m \to \mathcal{S}_m$ in the following way. If $(z,w) \in \pi_1^{-1}(P_j) \cap \mathcal{S}_m$ then

$$G(z,w) := \pi_2^{-1}|_{\pi_1^{-1}(P_j)}(\pi_2(z,w) - m).$$

Similarly if $(z,w) \in \pi_1^{-1}(Q_j) \cap \mathcal{S}_m$. One can easily check that G is a well defined holomorphic map which can be extended to all of $\mathcal{S}_m$ as an automorphism.

The upshot is to show that F is topologically conjugated to G on S^r_m, which will imply that f is topologically conjugated to $g := \pi_1 \circ G \circ \pi_1^{-1}$ on $\mathbb{C}^*_r$. Since also $\zeta \mapsto \zeta(1+\zeta^m)$ is topologically conjugated to g this will prove the theorem.

To this aim we define a C^∞ diffeomorphism $K : \mathcal{S}_m^r \to \mathcal{S}_m$ by gluing together F and G in such a way that $K = F$ outside some large compact subset of $\mathcal{S}_m$ and $K = G$ on a open set contained in such compact subset. Notice that this is possible for $|F(z, z^{-m}) - G(z, z^{-m})|$ goes to zero as $|z| \to \infty$. It is now enough to show that K is topologically conjugated to G.

The idea is to define a conjugation H on a set E, called *exaggerated fundamental domain*, such that for any $p \in \mathcal{S}_m$ there exists $a \in \mathbb{Z}$ such that $G^a(p) \in E$, and then extend the conjugation by means of the relation $H \circ G \circ H^{-1} = K$. The set E can be defined taking the set B of points where $K = G$ union $2m$ semi-strips from B to infinity delimited on each chart $\pi_2^{-1}(\pi_1(P_j)) \cap \mathcal{S}_m$ by $L_j = \pi_2^{-1}(\{\mathsf{Re}\,\zeta = 0\})$ and $G(L_j)$. Then H can be defined on E by means of $H|_B = \mathrm{id}$, $H|_{L_j} = \mathrm{id}$, $H|_{G(L_j)} = K(L_j)$ and glue together as a C^∞ diffeomorphism on each semi-strip. One can then check that E is absorbing iterates of G and thus H can be extended as wanted. □

The above proof shows that, if $f^n(\zeta) \neq \zeta$, then actually f is C^∞-conjugated to $\lambda\zeta(1 + \zeta^{kn})$ outside 0. One might suspect that with some more refinement it would be possible to extend the conjugation in (at least) a C^1-way to 0. However this is not the case, as shown by Martinet and Ramis [36]. In such a paper they provide a differentiable classification of parabolic germs. In particular they prove

Theorem 1.5 (Martinet-Ramis). *Let f and g be two parabolic germs at 0.*

(1) *If f and g are formally conjugated then they are topologically conjugated.*
(2) *If f and g are C^1-conjugated then they are holomorphically conjugated.*

The first statement is not surprising after Theorem 1.3 and the formal classification due to Voronin [54]. However the second result is very impressive!

Theorem 1.5 is actually a corollary of the holomorphic classification of parabolic germs which is also provided in [36]. This latter is also due to Voronin [54] and Ècalle [25], see also Il'yashenko [32]. Such a classification is quite complicated. A parabolic germ f is associated to an *orbits space* $\mathcal{F}_f$. Such $\mathcal{F}_f$ is a complex Riemann surface given by the amalgamated sum of $2m$ Riemann spheres. Each sphere represents a petals of f and the sum is defined by means of the behavior of f on the intersection of two consecutive petals (one attractive and the other repelling). The orbit spaces $\mathcal{F}_f$ provide the searched holomorphic invariants. See [36] for details.

1.3. **Elliptic case: $|\lambda| = 1$, $\lambda = e^{i\theta}$ for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$.** This case can be considered as a big world by itself, formed by several interesting problems—some still open—known as *small divisors problems*, related to physics and celestial mechanics. We only provide some small survey on the basic results.

Firstly, from a formal point of view one can kill all the terms after the linear one, so that $f(\zeta) = \lambda\zeta + O(\zeta^2)$ is *always* formally conjugated to $\zeta \mapsto \lambda\zeta$.

As for the holomorphic and topological linearization we have

Theorem 1.6. *Let f be an elliptic germ. Then f is holomorphically conjugated to $\zeta \mapsto \lambda\zeta$ if and only if the sequence $\{f^{\circ k}\}$ is uniformly bounded near 0. In particular f is holomorphically linearizable if and only if it is topologically linearizable.*

Proof. One direction is clear. Conversely, assume that $\{f^{\circ k}\}$ is uniformly bounded near 0. Let $\sigma_n(\zeta) = 1/n \sum_{j=0}^{n-1} \lambda^{-j} f^{\circ j}(\zeta)$. Then $\sigma_n \circ f = \lambda\sigma_{n+1} + O(1/n)$ and $\{\sigma_n\}$ is a normal family near 0. Therefore, up to subsequences, σ_n converges to a holomorphic map conjugating f to its differential. Finally, it is obvious that if f is topologically linearizable then $\{f^{\circ k}\}$ is uniformly bounded near 0 and thus f is also holomorphically linearizable. □

The question is whether all elliptic germs are holomorphic linearizable. The answer is known to be negative, and first examples where produced by Cremer. Indeed we have

Theorem 1.7 (Cremer)**.** *Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. If $\limsup_{n\to\infty} |\{n\theta\}|^{-1/n} = \infty$ (where $\{x\} = x - [x]$ with $[x]$ denoting the integral part of x) then there exists an elliptic germ $f(\zeta) = e^{i\theta}\zeta + O(\zeta^2)$ which is not linearizable.*

A number θ satisfying the condition of Theorem 1.7 is called a *Cremer number*. Cremer's number form a dense subset of $\mathbb{R}$ of zero Lebesgue measure. If an elliptic germ f is non-linearizable at 0, we say that 0 is a *Cremer point* for f.

On the other hand, sufficient arithmetic conditions on θ for $f(\zeta) = e^{i\theta}\zeta + \ldots$ to be linearizable were first given by Siegel. Thus we say that 0 is a *Siegel point* for f provided f is linearizable at 0. However Siegel's original conditions were not sharp. Later Bryuno gave a better sufficient condition on θ for f to be linearizable, and Yoccoz showed the necessity of such a condition. We suggest the interested reader to read, *e.g.*, the notes [35]. Here we content ourselves to state the result as follows:

Theorem 1.8 (Bryuno-Yoccoz)**.** *Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Let $\{p_n/q_n\}$ be the sequence of rational approximation to θ given by its continued fraction expansion. Then 0 is a Siegel point for all $f(\zeta) = e^{i\theta}\zeta + O(\zeta^2)$ if and only if*

$$\sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n} < \infty.$$

Notice that the numbers θ for which the condition stated in Theorem 1.8 is satisfied form a full Lebesgue measure subset of $\mathbb{R}$.

For what dynamics concerns, Siegel points are easily understood. Instead Cremer points are still quite mysterious, despite the remarkable work of R. Perez-Marco (see [40] and [41]). To state some of his results, we recall that a *small cycle* for f is a finite orbit of f, *i.e.*, a set $\{p_1, \ldots, p_n\} \subset \mathbb{C}^*$ such that $p_j \neq p_k$ and $f(p_j) = p_{j+1}$ modulo n. We say that a germ f has the *small cycles property* if for any open neighborhood U of 0 there exists a small cycle for f contained in U. If f has the small cycles property then small cycles accumulate at 0. Notice that an elliptic germ with the small cycles property is necessarily non-linearizable.

Theorem 1.9 (Perez-Marco). *There exist elliptic germs with the small cycles property. Not all non-linearizable elliptic germs have the small cycle property.*

Actually Perez-Marco provides a precise arithmetic condition on θ in order to decide whether the non-linearizable germ has the small cycles property. See [40] for details.

As far as we know, there is no topological nor holomorphic classification available for elliptic germs at Cremer points.

2. LOCAL DYNAMICS IN HIGHER DIMENSION

In higher dimension the situation is much more complicated than in dimension one. Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a germ of holomorphic diffeomorphism at the origin O fixing O. Even in several variables the spectrum of dF_O gives a first picture of the dynamics. However, several new phenomena may occur. First, dF_O may not be diagonalizable. This is mainly a technical problem which for simplicity we do not discuss here, so from now on we assume dF_O is diagonal with eigenvalues $\lambda_1, \ldots, \lambda_n$. Secondly the eigenvalues $\lambda_1, \ldots, \lambda_n$ might have *resonances*.

Definition 2.1. We say that the eigenvalue λ_s with $s \in \{1, \ldots, n\}$ is *resonant* if there exist $m_1, \ldots, m_n \in \mathbb{N}$ such that $m_1 + \ldots + m_n \geq 2$ and

$$\lambda_s = \lambda_1^{m_1} \cdots \lambda_n^{m_n}.$$

The vector $(m_1, \ldots, m_n)$ is said the *order* of resonance.

Notice that for the same eigenvalue there might be several order of resonances. Roughly speaking the eigenvalue λ_s is resonant if the dynamics along the other directions enter to disturb the dynamics along the eigendirection relative to λ_s.

As we saw in the previous section, in dimension one the only resonant case is the parabolic case and it is the only case where there is no formal linearization. So we start to study linearization and resonances.

2.1. Resonances and linearization.

We begin with a definition. Write $F = (F_1, \ldots, F_n)$, with series expansion $F_j = P_1^j + P_2^j + \ldots$ with P_k^j homogeneous polynomial in $z_1, \ldots, z_n$ of degree k. We denote by $P^j_{h_1,\ldots,h_m}$ the monomial $z_1^{h_1} \cdots z_n^{h_n}$ in $P^j_{h_1+\ldots+h_n}$. Assume dF_O has eigenvalues $\lambda_1, \ldots, \lambda_n$.

Definition 2.2. If λ_s is a resonant eigenvalue with order of resonance $(m_1, \ldots, m_n)$ we call $P^s_{m_1,\ldots,m_n}$ a *resonant monomial.*

With these definition we have

Theorem 2.3 (Poincaré-Dulac Normal Form). *Let F be a germ of holomorphic diffeomorphism of $\mathbb{C}^n$ fixing O. Assume that dF_O is diagonal. Then F is formally conjugated to a formal series $\hat{F} = dF_O + \hat{P}_2 + \ldots$, where the $\hat{P}_j$'s are polynomial made only of resonant monomials of F. In particular if dF_O has no resonances then F is formally linearizable.*

We note that the formal series provided by Theorem 2.3 is not unique in general. Such a series is called a *formal normal form* of F.

Proof of Theorem 2.3. First we try to kill the terms in P_2 using a transformation of type $T(z) = z + H(z)$ with H a polynomial of degree 2. Thus

$$T \circ F \circ T^{-1}(z) = dF_O(z) + P_2(z) + H \circ dF_O(z) - dF_O \circ H(z) + O(\|z\|^3).$$

To kill P_2 one has to solve the so-called *homological functional equation* in H given by

$$dF_O \circ H - H \circ dF_O = P_2.$$

This can be always solved provided there are no resonances of order 2. Otherwise the resonant terms might survive. Keeping on solving homological equation of increasing degree one has the statement. □

The question is then when the formal change of variables provided by Theorem 2.3 is actually convergent. The answer is provided by Poincaré himself.

Definition 2.4. We say that F belongs to the *Poincaré domain* at 0 if all the eigenvalues of dF_O have modulus strictly less than 1 or they all have modulus strictly greater than 1. Otherwise we say that F belongs to the *Siegel domain* at O.

Thus we have (see, *e.g.*, [7] for a proof):

Theorem 2.5 (Poincaré-Dulac)**.** *If F belongs to the Poincaré domain at O then F is holomorphically conjugated to a polynomial normal form. In particular if F has no resonances at O then it is holomorphically linearizable.*

Using this result, Reich [44], [45] gave the holomorphic classification of germs in the Poincaré domain at O.

If F belongs to the Siegel domain one may also ask for linearization or convergence of the formal change of variables in Theorem 2.3. The first result in this direction (and the reason for naming Siegel in this context) is due to Siegel (see, *e.g.* [7]). We state it here as follows:

Theorem 2.6 (Siegel)**.** *Let F be a germ of holomorphic diffeomorphism fixing O. Let denote by $\{\lambda_1, \ldots, \lambda_n\}$ the eigenvalues of dF_O. If there exist $C > 0$ and $\nu \in \mathbb{N}$ such that for all $s = 1, \ldots, n$ and $m_1, \ldots, m_n \in \mathbb{N}$ such that $\sum m_j \geq 2$ and $|\lambda_s - \lambda_1^{m_1} \cdots \lambda_n^{m_n}| \neq 0$ it holds*

$$|\lambda_s - \lambda_1^{m_1} \cdots \lambda_n^{m_n}| \geq \frac{C}{(\sum_{j=1}^n m_j)^\nu} \tag{2.1}$$

then F is holomorphically conjugated to a normal form. In particular if F satisfies (2.1) *and F is formally linearizable (for instance if F has no resonances at O) then F is holomorphically linearizable.*

It is worth noticing that the condition in Theorem 2.6 is full Lebesgue measure for ν sufficiently big. Thus, collecting all the previous results, roughly speaking, we can say that almost all germs of holomorphic diffeomorphism fixing O are holomorphically linearizable.

It should also be remarked that Bryuno [16] gave an improvement of Siegel's condition of Theorem 2.6.

In several variables it makes also sense to ask for *partial linearization*, or linearization along some submanifold. Namely, the full germ F might not be linearizable, but it may exist a complex submanifold M passing through O such that $F(M) \subset M$ and $F|_M$ is linearizable. Again, partial linearization depends on small divisors.

Theorem 2.7 (Pöschel)**.** *Let F be a germ of holomorphic diffeomorphism fixing O. Let denote by $\{\lambda_1, \ldots, \lambda_n\}$ the eigenvalues of dF_O with eigenspaces $E(\lambda_j)$. Let $k \leq n$. If there exist $C > 0$ and $\nu \in \mathbb{N}$ such that for all $s = 1, \ldots, n$ and $m_1, \ldots, m_k \in \mathbb{N}$ such that $\sum m_j \geq 2$ it holds*

$$|\lambda_s - \lambda_1^{m_1} \cdots \lambda_k^{m_k}| \geq \frac{C}{(\sum_{j=1}^k m_j)^\nu} \tag{2.2}$$

then there exists an F-invariant complex submanifold $M \subset \mathbb{C}^n$ such that $T_O M = \sum_{j=1}^k E(\lambda_j)$ and $F|_M$ is holomorphically linearizable.

Again, condition (2.2) can be improved, see [43].

2.2. **Stable/unstable center manifolds.** Assume that F is linearizable by means of the conjugation φ, *i.e.*, $\varphi \circ F \circ \varphi^{-1} = dF_O$. If E is an eigenspace of dF_O of dimension k then $\varphi(E)$ is a complex submanifold of dimension k of $\mathbb{C}^n$ containing O and which is F-invariant. Moreover the action of F on E is essentially determined by the eigenvalue—say λ—associated to E. With obvious meaning, the manifold $\varphi(E)$ is called *stable* if $|\lambda| < 1$, *unstable* if $|\lambda| > 1$ and *central* if $|\lambda| = 1$.

Now, linearizable germs are dense in the space of germs (with any decent topology, for instance the compact-open topology). Thus one might hope to recover stable/unstable and central manifolds even in the non-linearizable case. This is however only partially true. To fix notations, let E_s be the sum of eigenspaces of dF_O associated to eigenvalues of modulus strictly less than 1. Let E_u be the sum of eigenspaces of dF_O associated to eigenvalues of modulus strictly larger than 1. Finally let E_c be the sum of eigenspaces of modulus 1. Then the stable/unstable center manifold is the following:

Theorem 2.8 (Stable/Unstable Center Manifolds)**.** *Let F be a germ of holomorphic diffeomorphism at O fixing O.*

(1) *There exists a unique F-invariant complex submanifold $W_s \subset \mathbb{C}^n$ of dimension $\dim_{\mathbb{C}} E_s$ such that $O \in W_s$, $T_O W_s = E_s$, and $F^{\circ k}(p) \to O$ as $k \to \infty$ for all $p \in W_s$.*
(2) *There exists a unique F-invariant complex submanifold $W_u \subset \mathbb{C}^n$ of dimension $\dim_{\mathbb{C}} E_u$ such that $O \in W_u$, $T_O W_u = E_u$, and $F^{\circ -k}(p) \to O$ as $k \to \infty$ for all $p \in W_u$.*
(3) *There exists a (not unique) F-invariant C^∞ submanifold $W_c \subset \mathbb{C}^n$ of dimension $\dim_{\mathbb{R}} E_c$ such that $O \in W_c$ and $T_O W_c = E_c$.*

Notice that $F|_{W_s}$ and $F|_{W_u}$ are holomorphically conjugated to a polynomial normal form by Theorem 2.5.

Theorem 2.8 is not the most general statement one can get. For instance, one can prove the existence of complex stable/unstable manifolds related to any eigenspace associated to an eigenvalue of modulus strictly smaller/larger than 1. Moreover one can give several useful characterization of W_s, W_u, W_c. For these and for proofs, we refer the interested reader to [31] or [3], where also the non-invertible and non-local cases are considered. The theorem is originally due to Pesin, Hadamard and Wu [56] for the complex category.

It is important to note that in general the non-uniqueness of W_c prevents this latter to have a complex structure.

2.3. **Hyperbolic case.** We say that O is a *hyperbolic point* for F if dF_O does not have eigenvalues of modulus 1. In this case Theorem 2.8 gives a clear picture of the dynamics near O, for no center manifolds appear.

If F is in the Poincaré domain at O (that is all the eigenvalues have modulus either strictly smaller than 1 or strictly greater than 1) Theorem 2.8 assures that all points in an open neighborhood of O are attracted to O by F or by F^{-1}.

If F has some eigenvalues of modulus > 1 and some of modulus < 1 then Theorem 2.8 gives two F-invariant complex submanifold W_s, W_u where the dynamics is attractive/repulsive. Any other point in a neighborhood of O escapes from O both iterating forward and iterating backward, exactly as if F were linearizable. Indeed hyperbolic germs are topologically linearizable:

Theorem 2.9 (Gröbman-Hartman)**.** *If F is a germ of hyperbolic holomorphic diffeomorphism at O fixing O then F is topologically linearizable at O.*

Aside the original references, see [3] for a proof.

2.4. **Parabolic cases.** A germ of diffeomorphism F at O fixing O is *parabolic* if at least one of the eigenvalues of dF_O is a root of unity. This terminology is not standard since the study of holomorphic dynamics in several dimension is only at the beginning. Also, some results are true for dimension two, while they are false or unknown for dimension greater than 2.

2.4.1. *Semi-attractive case.* We say that a parabolic germ F is *semi-attractive* if 1 is an eigenvalue of dF_O and all the other eigenvalues have modulus strictly less than 1 (if all the other eigenvalues have modulus strictly greater than 1 we argue on F^{-1}). There are essentially two cases to be distinguished here: F has or not a submanifold of fixed points. In case F has a submanifold of fixed points (of the right dimension) we have a result due to Nishimura [38] which roughly speaking says that, in absence of resonances, F is conjugated along S to its action L_F on the normal bundle N_S to S in $\mathbb{C}^n$. The precise result is:

Theorem 2.10 (Nishimura)**.** *Let F be a parabolic germ at O and assume there exists a submanifold $S \subset \mathbb{C}^n$ such that $O \in S$ and $F|_S = \mathrm{id}$. Let $\{1, \lambda_1(p), \ldots, \lambda_m(p)\}$ be the eigenvalues of dF_p at $p \in S$. Assume that for any $p \in S$, T_pS is the eigenspace related to 1, $|\lambda_j(p)| < 1$ for $j = 1, \ldots, m$ and there are no resonances among $\lambda_1(p), \ldots, \lambda_m(p)$. Then there exists an open neighborhood U of S and a unique biholomorphic map $\varphi : N_S \to U$ such that $F \circ \varphi = \varphi \circ L_F$.*

Other results of more global nature (obtained from the local situation by means of blow-ups) are contained in [6], we come back on these later when talking about germs tangent to identity.

In case F has no curves of fixed points there are results of Ueda [52], [53], Hakim [29] and Rivi [46] which generalize older results of Fatou [27]. Such results essentially state that, under suitable generic hypotheses, there exist "fatty petals" (called *parabolic manifolds* or *basins of attraction* when they have dimension n) for F at O. To be more precise,

Definition 2.11. A *parabolic manifold* M for F at O is an F-invariant complex submanifold of $\mathbb{C}^n$ containing O on the boundary such that for any $p \in M$ the sequence of iterates $\{F^{\circ k}(p)\}$ converges to O.

Roughly speaking, the number of parabolic manifolds is related to the "order" of $F - \mathrm{id}$ along the parabolic direction at O while their dimension is given by the number of non-unimodular eigenvalues of dF_O. Here we content ourselves to state the following result:

Theorem 2.12 (Hakim). *Let F be a semi-attractive parabolic germ at O, with 1 as eigenvalue of dF_O of (algebraic) multiplicity 1. If O is an isolated fixed point of F then there exist k disjoint basins of attraction for F at O, where $k + 1 \geq 2$ is the "order" of $F - \mathrm{id}$ at O.*

It is worth noticing that if F is an automorphism of $\mathbb{C}^2$ then each basin of attraction provided by Theorem 2.12 is biholomorphic to $\mathbb{C}^2$ (the existence of proper subsets of $\mathbb{C}^n$ biholomorphic to $\mathbb{C}^n$ for $n > 1$ is known as the *Fatou-Bierbach phenomenon*).

Ueda, whose works hold in $\mathbb{C}^2$, provided precise information on the shape of the basin of attraction (in case the order of $F - \mathrm{id}$ is exactly 2) and showed that F is conjugated to $(z, w) \mapsto (z + 1, w)$ on such a basin of attraction.

Rivi generalizes Theorem 2.12 under the hypothesis that 1 has algebraic multiplicity greater than 1, proving that *generically* there exist parabolic manifolds for F at O (here the word "generically" refers to the existence of "non-degenerate characteristic directions" which we will discuss later for germs tangent to the identity).

As for the topological classification of semi-attractive germs, we have the following result in $\mathbb{C}^2$ due to Canille-Martins [20].

Theorem 2.13 (Canille-Martins). *Let F be a semi-attractive germ of $\mathbb{C}^2$ fixing O. Then there exists $k \in \mathbb{N}$ such that F is topologically conjugated to the map $(z, w) \mapsto (z + z^k, 1/2w)$.*

Sketch of the Proof. By Theorem 2.8 there exists a real differentiable two dimensional F-invariant manifold M passing through O and tangent to the eigenspace of 1 at O. Such M is not unique. However by the theory of normal hyperbolic system of Palis and Takens [39] the dynamics from a topological point of view of F near O depends only on the dynamics of F on M. If it happens that M is complex then $F|_M$ is topologically conjugated to $z \mapsto z + z^k$ by Theorem 1.3. If M does not have a complex structure then the result is still true using the theory of real diffeomorphisms of [24]. □

It is clear that if F is a parabolic germ such that $F^{\circ k}$ is semi-attractive for some $k \in \mathbb{N}$ then the previous results apply to $F^{\circ k}$ and from this one recovers information on the dynamics of F. We left details to the reader.

2.4.2. *Non-attractive case.* We say that a parabolic germ is *non-attractive* if all eigenvalues of dF_O have modulus 1.

Let write the spectrum of dF_O as $R \cup I$, where R contains the roots of unity and I the other unimodular eigenvalues. The dynamics along the eigenspaces related to I are described by Theorem 2.7 in absence of small divisors.

The present section deals with dynamics along the directions related to the eigenvalues in R, and thus, up to replacing F with some higher iterate, along the eigenspace relative to 1. We start with the following lemma

Lemma 2.14 (Hakim, Abate, Bracci-Molino)**.** *Let F be a parabolic non-attractive germ of $\mathbb{C}^2$ fixing O. If*

$$F(z,w) = (z + z^k + O(z^{k+1}, z^k w), \lambda w - \delta z^{k-1} w + O(z^{k+2}, z^k w))$$

with $\mathsf{Re}\,(\delta\overline{\lambda}) < 0$, then there exist $k-1$ parabolic curves (ı.e. parabolic manifolds of complex dimension 1) for F at O tangent to $[1:0]$.

Here we say that a parabolic curve P is tangent to $[a,b]$ whether the complex span of the tangent cone of P at O is generated by (a,b).

The previous result is due (in more than two variables) to Hakim [30] and Abate [2] for $\lambda = 1$, and to Molino and the author [11] for $\lambda \neq 1$ (and $|\lambda| = 1$).

Very rough sketch of the Proof of Lemma 2.14. If the second component of F has no pure terms in z then the curve $\{w = 0\}$ is F-invariant and the result follows from Theorem 1.2. In general, since the pure terms in z in the second component have sufficiently high order, one can infer that parabolic curves—if any—should not be too far (in an appropriate topology) from the petals we would have in case the second component of F were divisible by w, *i.e.*, the petals of $z \mapsto F_1(z,0)$. Thus we may try to find the parabolic curves among those of the form $\zeta \mapsto (\zeta, \zeta^2 u(\zeta))$ for ζ belonging to a petal of $F_1(z,0)$ and $|u|_\infty < \infty$. These curves form a Banach space (with norm given by the L^∞ norm of u). Starting from F one can define an operator on such a Banach space whose fixed points are exactly the searched parabolic curves. Then one shows that such an operator is a contraction and the fixed point theorem provides then the existence of a fixed point. □

In principle Lemma 2.14 is a powerful tool. Given a parabolic non-attractive germ F of $\mathbb{C}^2$, if it is possible to change coordinates in such a way that F has the wanted form then F has a certain number of parabolic curves at O tangent to the eigenspace of 1. Moreover, one can allow also "meromorphic changes of variables".

To be more precise, let $\pi : \tilde{\mathbb{C}}^2 \to \mathbb{C}^2$ be the *blow-up* (also called quadratic transformation) at O. Then it is possible to define a germ of holomorphic diffeomorphism $\tilde{F}$ near the exceptional divisor $D := \pi^{-1}(O)$ such that $\pi \circ \tilde{F} = F \circ \pi$ and $\tilde{F}|_D([v]) = dF_O([v])$, for all $[v] \in D \simeq \mathbb{CP}^1$ (see, *e.g.*, [1]). It is clear that if P is a parabolic curve for $\tilde{F}$ at a point $[v] \in D$ then $\pi(P)$ is a parabolic curve for F at O tangent to the direction v.

Then, if after changes of coordinates and/or blow-ups one finds that F (or its blow-up) has the form required by Lemma 2.14 it follows that F has parabolic curves.

In practice however it is almost impossible to explicitly perform holomorphic changes of coordinates or blow-ups in order to obtain that the germ has the form required in Lemma 2.14. Nonetheless, what one does in practice is to find some invariant, easily to be computed, attached to F which tells whether F has the wanted form after changes of variables or blow-ups.

In [11] two (holomorphic and formal) invariants are defined for the case $\lambda \neq 1$. To describe them the authors introduced a generalization of the Poincarè-Dulac normal forms, called *ultra-resonant normal forms*. These latter are somewhat better than the usual Poincaré-Dulac normal forms because the existence of a convergent ultra-resonant normal form is related to the existence of a curve of fixed points for F. However, for what the definition of invariants concern, we can also use Poincaré-Dulac normal forms. So, let $\hat{F}$ be a Poincaré-Dulac normal form for F. Without loss of generality we can assume that

$$\hat{F}(z,w) = \left(z + \sum_{j+k\geq 2} p_{j,k} z^j w^k, \lambda w + \sum_{j+k\geq 2} q_{j,k} z^j w^k\right).$$

We let

$$\nu(F) := \min\{j \in \mathbb{N} : p_{j,0} \neq 0\}, \quad \mu(F,w) := \min\{j \in \mathbb{N} : q_{j,1} \neq 0\}.$$

If $\nu(F) < \infty$, we let $\Theta(F) := \nu(F) - \mu(F,w) - 1$ (with the convention that $\Theta(F) = -\infty$ if $\mu(F,w) = \infty$). We say that F is *dynamically separating* if $\nu(F) < \infty$ and $\Theta(F) \leq 0$.

One should prove that $\nu(F)$ and being dynamically separating are definitions well-posed, since as already remarked, Poincaré-Dulac normal forms are by no means unique. This can be done as in [11]. Let us only note that $\nu(F)$ can be viewed as the "order" of F on the formal curve of its fixed points. Indeed, the formal Poincaré-Dulac normal form has no pure terms in z in the second component, and thus $\{w = 0\}$ is a "curve of fixed points" for $\hat{F}$.

We said before that invariants should be quite easy to be computed, while, finding a Poincaré-Dulac normal form might not be so easy. Actually, to define $\nu(F)$ and see whether F is dynamically separating one needs only to solve some homological equations as in the proof of Theorem 2.3 until the first non-zero pure term in z in the second component of F has degree equal or greater than the first non-zero pure term in z in the first component of F. For instance, if

$$F(z,w) = (z + az^2 + O(z^3, zw, w^2), \lambda w + O(z^2, zw, w^2))$$

for some $a \neq 0$ then $\nu(F) = 2$ and F is dynamically separating. For dynamically separating maps one can perform changes of coordinates and blow-ups to obtain the form needed in Lemma 2.14. Thus we have:

Theorem 2.15 (Bracci-Molino). *Let F be a parabolic germ of $\mathbb{C}^2$ at O such that dF_O has eigenvalues $\{1, \lambda\}$ with $|\lambda| = 1$ and $\lambda \neq 1$. If F is dynamically separating at O then there exist $\nu(F) - 1$ parabolic curves for F at O tangent to the eigenspace of 1.*

It is likely that a result similar to Theorem 2.15 holds in $\mathbb{C}^n$ for $n > 1$.

We turn now our attention to the case of non-attractive germs tangent to the identity, *i.e.*, such that $dF_O = \mathrm{id}$. These are, up to now, the most studied for some unexpected beautiful geometry that can be found inside.

In the preliminary work [55], Weickert constructs a family of automorphisms of $\mathbb{C}^2$ tangent to the identity at O with a basin of attraction at O, biholomorphic to $\mathbb{C}^2$ on which the automorphisms are conjugated to the map $(z, w) \mapsto (z + 1, w)$.

In his huge work [25] (see also [26]), Écalle gives a (partial) formal classification of germs tangent to the identity, proving as an intermediate step that "generically" a germ tangent to the identity has a certain number of parabolic curves. His proof is based on the theory of resurgence, a very elaborate tool. Recently, Hakim [30] gave a complete analytic proof of such a result. To better describe her approach we need some definitions. To avoid triviality, we always suppose $F \neq \mathrm{id}$, even if not explicitly stated.

Definition 2.16. Let F be a germ of $\mathbb{C}^n$ fixing O and tangent to the identity at O. Let $F(X) = X + P_h(X) + \ldots$, $h \geq 2$ be the expansion of F in homogeneous polynomials, $P_h(X) \neq 0$. The polynomial $P_h(X)$ is called the *Hakim polynomial* and the integer h the *order* of F at O.

Let $v \in \mathbb{C}^n$ be a nonzero vector such that $P_h(v) = \alpha v$ for some $\alpha \in \mathbb{C}$. Then v is called a *characteristic direction* for F. If moreover $\alpha \neq 0$ then v is said a *nondegenerate characteristic direction.*

It can be proved that if P is a parabolic curve for F at O tangent to v then v is a characteristic direction. However there exist examples of germs tangent to the identity with a parabolic curve not tangent to a single direction (that is with tangent cone spanning a vector space of dimension greater than one). Hakim's (and Écalle's) result is the following:

Theorem 2.17 (Écalle, Hakim). *Let F be a germ of holomorphic diffeomorphism of $\mathbb{C}^n$ fixing O and tangent to the identity at O with order h. If v is a nondegenerate characteristic direction for F then there exist (at least) $h - 1$ parabolic curves tangent to v.*

The proof is essentially the one given for Lemma 2.14: with a finite number of blow-ups and changes of coordinates one obtain a "good form" for F (or its blow-up)—just like the one written in Lemma 2.14—and then can argue similarly. However it should be notice that in general (namely if one of the eigenvalues to be introduced in Theorem 2.18 is a natural number), the transformations involved are much more complicated!

Actually Hakim's work provides the existence of basins of attraction or parabolic manifolds according to other invariants related to any nondegenerate characteristic direction. Let v be a nondegenerate characteristic direction for F and let P_h be the Hakim polynomial. We denote by $A(v) := d(P_h)_{[v]} - \mathrm{id} : T_{[v]}\mathbb{CP}^{n-1} \to T_{[v]}\mathbb{CP}^{n-1}$. Then we have

Theorem 2.18 (Hakim)**.** *Let F be a germ of holomorphic diffeomorphism of $\mathbb{C}^n$ fixing O and tangent to the identity at O. Let v be a nondegenerate characteristic direction. Let $\beta_1, \ldots, \beta_{n-1} \in \mathbb{C}$ be the eigenvalues of $A(v)$. Moreover assume $\mathsf{Re}\,\beta_1, \ldots, \mathsf{Re}\,\beta_m > 0$ and $\mathsf{Re}\,\beta_{m+1}, \ldots, \mathsf{Re}\,\beta_{n-1} \leq 0$ for some $m \leq n-1$ and let E be the sum of the eigenspaces associated to $\beta_1, \ldots, \beta_m$. Then there exists a parabolic manifold M of dimension $m+1$ tangent to $\mathbb{C}v \oplus E$ at O such that for all $p \in M$ the sequence $\{F^{\circ k}(p)\}$ tends to O along a trajectory tangent to v.*

In particular if all the eigenvalues of $A(v)$ have positive real part then there exists a basin of attraction for F at O.

Hakim's results, and the fact that there are examples of germs tangent to the identity with no nondegenerate characteristic directions give rise to the question: is it true that *all* germs tangent to the identity do have parabolic curves?

The answer is positive in dimension two and it was solved by Abate [2], while it is presently unknown in dimension greater than two. We have:

Theorem 2.19 (Abate)**.** *Let F be a germ of holomorphic diffeomorphism of $\mathbb{C}^2$, having O as an isolated fixed point, and tangent to the identity at O. Then there exists at least one parabolic curve for F at O.*

The original proof of Abate—while correct—is quite mysterious. In [9] and [6] we gave a different explanation, based on the better known theory of holomorphic foliations. Therefore, in order to provide details we need to recall some basic facts on the local theory of holomorphic foliations.

Interlude on holomorphic foliations

A *local (one dimensional) holomorphic foliation* $\mathcal{F}$ in $\mathbb{C}^n$ at O is roughly speaking the data of a germ of a holomorphic vector field at O up to nonzero multiples. More precisely, $\mathcal{F}$ is given by a holomorphic line bundle L near O and a morphism of vector bundle $\varphi : L \to T\mathbb{C}^n$. If 1 is a base frame of L near O then $v = \varphi(1)$ is a vector field. The *singularities* of $\mathcal{F}$ are defined to be the points where φ is zero, or, equivalently the points where $v = 0$. A *leaf* of $\mathcal{F}$ is an integral curve of v, regardless of its parameterization. Namely, a (possibly singular) curve S is a leaf of $\mathcal{F}$ if the vector defining $\mathcal{F}$ belongs to the (Zariski) tangent space of S at all points of S.

In case O is not a singularity of $\mathcal{F}$ then the well-known Cauchy-Kowaleskaya Theorem provides a *unique* non-singular leaf for $\mathcal{F}$ at O. Moreover, since singularities are closed, one can choose local coordinates $\{z_1, \ldots, z_n\}$ in such a way that $\mathcal{F}$ is generated by $\frac{\partial}{\partial z_1}$ ("linearization" of the foliation).

The problem is when O is a singularity of $\mathcal{F}$. We are mainly interested in the case where $\mathcal{F}$ has an isolated singularity at O. The constant "curve" O is clearly a "leaf" of $\mathcal{F}$. However it is not unique in general. For instance, if $\mathcal{F}$ is generated by $\sum_{j=1}^n z_j \frac{\partial}{\partial z_j}$ then all complex lines through O are leaves for $\mathcal{F}$. The study of the leaves of a holomorphic foliation is the subject of the *holomorphic continuous dynamics*.

There are strict relations between continuous and discrete dynamics. A first way is provided by associating to a vector field its time one flow, which is a diffeomorphism of $\mathbb{C}^n$. The problem with this is that the converse operation is not always (actually seldom) possible in the holomorphic category. That is to say, starting from a holomorphic vector field, the associated flow is holomorphic, but conversely, starting from a holomorphic diffeomorphism there are in general no holomorphic vector fields whose time one flow coincides with the given diffeomorphism. This operation can be done (locally) only in the formal category, using the so called Campbell-Hausdorff formula. Nonetheless this is the philosophical argument which provides a strict link between continuous and discrete dynamics.

A second way to relate continuous and discrete dynamics is by means of the *holonomy* or *Poincaré return map*. In our case, in presence of an isolated singularity for a germ of vector field of $\mathbb{C}^n$ at O and a nonsingular simple-connected leaf P passing through O, the holonomy is a germ of holomorphic diffeomorphism of $\mathbb{C}^{n-1}$ at O constructed as follows. Take a (germ of a) complex ($n-1$-dimensional) transverse T to P at a point $p \in P$ near to O. Let γ be a generator of the ciclic group $\pi_1(P \setminus \{0\}; p) \simeq \mathbb{Z}$. If $q \in T$, following the leaf of $\mathcal{F}$ starting from q which projects to γ we finish at some point $F(q) \in T$. The application $q \mapsto F(q)$ is a germ of holomorphic diffeomorphism of T at p (fixing p), called the *local holonomy* of $\mathcal{F}$ (the holonomy can be defined more generally for nonsingular foliations). The dynamical properties of F read the dynamics of $\mathcal{F}$. A custom result is that, in general, two foliations are the same from the topological point of view if and only if their holonomies are topological conjugated. This is particular useful for foliations of $\mathbb{C}^2$ for then the holonomy is a germ of diffeomorphism of $\mathbb{C}$. It is known after Perez-Marco and Yoccoz [42] that any germ of holomorphic diffeomorphism in $\mathbb{C}$ can be realized as the holonomy of a suitable germ of holomorphic foliation in $\mathbb{C}^2$.

There is a third way, much more easy to handle in practice, to relate a holomorphic germ of diffeomorphism to a (family of) holomorphic foliations, introduced in [9] by the author for dimension two and generalized to higher dimension in [6]. This will be discussed later to solve our problem about existence of parabolic curves.

Now, let $\mathcal{F}$ be a germ of holomorphic foliation with an isolated singularity at O. We ask for existence of curves through O which are leaves of $\mathcal{F}$. when they exist they are called *separatrices* for they "separate" the dynamics.

First we examine the case $\mathcal{F}$ is a germ of foliation in $\mathbb{C}^2$ with an isolated singularity at O. Using a process called "saturation" one can always assume that the subvariety of singularities of a holomorphic foliation have codimension 2, thus in dimension two it is not restrictive to impose that O is an isolated singularity.

Let $X = (\alpha x+\beta y+O(x^2, y^2, xy))\frac{\partial}{\partial\partial x}+(\gamma x+\delta y+O(x^2, y^2, xy))\frac{\partial}{\partial\partial y}$ be a holomorphic vector field representing $\mathcal{F}$. One first looks at the *linear part* of X defined by the linear transformation

$$(x, y) \mapsto J^1X(x, y) := (x, y) \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

This is not well defined in general for X is not uniquely attached to $\mathcal{F}$. However any other vector field associated to $\mathcal{F}$ is a (nonzero) multiple of X. Therefore, if λ_1, λ_2 are the eigenvalues of

J^1X, both are zero if and only if they are zero for all the vectors associated to $\mathcal{F}$. Moreover if $\lambda_2 \neq 0$ then the ratio λ_1/λ_2 is independent of the vector field chosen to represent $\mathcal{F}$. therefore we can well say

Definition 2.20. The singularity O of a holomorphic foliation in $\mathbb{C}^2$ is *reduced* if

$(\star_1)$ either $\lambda_2 \neq 0$ and $\lambda_1/\lambda_2 \notin \mathbb{Q}^+ \cup \{0\}$,
$(\star_2)$ or $\lambda_2 \neq 0$ and $\lambda_1 = 0$.

In the reduced cases normal simple forms were known since Poincaré and Dulac (see, *e.g.*, [37] or [17]). In particular from such normal forms we can infer

Theorem 2.21 (Poincaré-Dulac)**.** *If O is a reduced $(\star_1)$ singularity for $\mathcal{F}$ then there exist exactly two complex separatrices for $\mathcal{F}$, nonsingular at O, which intersect transversally at O.*

If O is a $(\star_2)$ reduced singularity for $\mathcal{F}$ then there exists one nonsingular separatrix for $\mathcal{F}$ at O. There is also a second formal separatrix at O which may or may not converge.

Therefore in the "generic" case of a reduced singularity Theorem 2.21 provides a positive answer to the question about the existence of other leaves. Somehow, this result can be considered the analogous of Theorem 2.17 for germs of diffeomorphisms tangent to the identity.

If O is not a reduced singularity one can try to blow-up the origin and blow-up the foliation $\mathcal{F}$. Thus, let $\pi : \tilde{\mathbb{C}}^2 \to \mathbb{C}^2$ be the blow-up at O. One can define a foliation $\tilde{\mathcal{F}}$ on $\tilde{\mathbb{C}}^2$ near the exceptional divisor $D = \pi^{-1}(O)$. A way to define $\tilde{\mathcal{F}}$ is to consider a holomorphic one-form ω such that $\omega(X) \equiv 0$, and say that $\tilde{\mathcal{F}}$ is defined by the saturated of the pull-back $\pi^*(\omega)$ (in local coordinates one obtains the saturated of $\pi^*(\omega)$ by dividing the coefficients by a defining equation of D at the highest possible power in order to have such new form holomorphic and with only isolated zeros).

If D is not a leaf of $\tilde{\mathcal{F}}$ then we say that the singularity O is *dicritical.* It is clear that, by the Cauchy-Kowaleskaya Theorem, for all but a finite number of points of D, there exists a nonsingular leaf for $\tilde{\mathcal{F}}$ which projects down to a leaf of $\mathcal{F}$. Therefore if O is dicritical there exist infinitely many separatrices through O.

Now assume that O is not dicritical. Then $\tilde{\mathcal{F}}$ has only finitely many singularities on D. The idea is that if some of them is not reduced one can continue the process of blow-ups to hope to reduce all the singularities. This is exactly the case

Theorem 2.22 (Saidenberg)**.** *Let $\mathcal{F}$ be a germ of holomorphic foliation of $\mathbb{C}^2$ at O with an isolated singularity at O. After a finite number of blow-ups one obtains a holomorphic foliation with only reduced singularities.*

A proof of Saidenberg theorem can be found in [17]. Notice that the theorem applies also to dicritical singularities, even if, from the point of view of existence of separatrices is not very interesting.

Even with the Saidenberg resolution of singularities Theorem one cannot conclude that there always exists a separatrix for $\mathcal{F}$. Indeed it could happen that all the reduced singularities of type $(\star_1)$ are corners of the exceptional divisor and singularities outside corners are all of type $(\star_2)$ with non-convergent formal separatrix. This is not the case, and the proof is based on the celebrated Camacho-Sad index theorem. In [19] Camacho and Sad proved the following theorem

Theorem 2.23 (Camacho-Sad)**.** *Let $\mathcal{F}$ be a holomorphic foliation on a complex two dimensional manifold and let $S \subset M$ be a nonsingular compact curve which is a leaf of $\mathcal{F}$. Then at all singularities of $\mathcal{F}$ on S it is possible to associate a complex number Res$(\mathcal{F}, S; p)$ such that*

$$\sum_{p \in \mathit{Sing}(\mathcal{F})} \mathit{Res}(\mathcal{F}, S; p) = S \cdot S.$$

Residues are strictly related to dynamics, and decrease by one after blow-ups. Using those properties, Camacho and Sad, with a complicated combinatorics, showed that after having reduced all singularities with Theorem 2.22 then there must be a $(\star_1)$-reduced singularity at a nonsingular point of the exceptional divisor, and thus the second separatrix given by Theorem 2.21 blows down to a (possibly singular) separatrix for the original foliation.

The combinatorics part in Camacho-Sad argument can be very much simplified, as done by Toma, Sebastiani and Cano, see, *e.g.*, [17] or [21] for details. Theorem 2.23 itself gave rise to lots of researches on "index theorems" and residues theory, especially by Lins Neto, Lehmann, Camacho, Suwa, Seade, Brunella, Brasselet, Abate, Tovena and the author (see [50] for a good account on residues theorems for foliations and, [6], [14], [13], [10] for residues theorems for diffeomorphisms and generalizations).

For germs of holomorphic foliations in $\mathbb{C}^n$ with $n > 2$ the existence of separatrices is a much more involved problem. Indeed there are examples without separatrices, due to Gomez-Mont and Luengo [28].

We go back to the problem of finding parabolic curves for germs of holomorphic diffeomorphisms tangent to the identity in $\mathbb{C}^2$. There is a philosophical explanation on the reason why there should always exist parabolic curves for holomorphic germs of diffeomorphisms. The argument goes like this. One can consider the germ F as the time one flow of a vector field X. Unfortunately, such a vector field is not holomorphic in general, but it is only formal. Nonetheless, one should argue as in the Camacho-Sad paper [19] in order to obtain a "formal separatrix" for X. Pieces of such separatrix should converge and give the searched parabolic curves for F. It is clear that there are several problems for making this argument precise, and also, even if one makes it work in this situation, such technique does not seem to be handleable in more general situation (like for instance germs on singular surfaces or with singular curves of fixed points). However it serves as a guide for what kind of results one might expect.

In [9], [6], [12] we introduced another method to relate diffeomorphisms to foliations, which seems to give interesting results. Let us roughly describe it. Let F be a germ of holomorphic diffeomorphism at O in $\mathbb{C}^2$. We consider a family of holomorphic foliations given by

$$\mathcal{F}_F^{z,w} = \{(z \circ F - z)\frac{\partial}{\partial z} + (w \circ F - w)\frac{\partial}{\partial w} \mid dz_O \wedge dw_O \neq 0\}.$$

Of course the foliation $\mathcal{F}_F^{z,w}$ depends on z, w. However it can be proved that if O is a singularity for one of such foliation it is so for all the others. Moreover, if O is a singularity of $\mathcal{F}_F^{z,w}$, the linear part of $\mathcal{F}_F^{z,w}$ is independent of z, w up to a non-zero multiple. In particular if O is a singularity for $\mathcal{F}_F^{z,w}$ one can define O as a singularity of reduced or non-reduced type for F according to the kind of singularity of $\mathcal{F}_F^{z,w}$ regardless of the z, w chosen. In particular one can define a *dicritical* point of F to be a point which is dicritical for $\mathcal{F}_F^{z,w}$. Also, Theorem 2.22, provides a theorem of reduction of singularities for F (already proved by direct methods in [2]). Therefore if we had a "residue theorem" like Theorem 2.23 for the blow-up of F on the exceptional divisor, with residues reading the dynamics, then we could argue as in [19] to prove the existence of parabolic curves.

We give here a version of the residue theorem needed, as obtained in [6].

Let M be a complex manifold of dimension n, $F : M \to M$ be a holomorphic map having a nonsingular compact hypersurface S as fixed points locus. It is possible to define a morphism, called the *canonical section*, of vector bundle

$$X_F : N_S^{\otimes \nu_F} \to TM|_S,$$

where ν_F is the "order of vanishing" of $F - \mathrm{id}$ on S. For instance if $\nu_F = 1$ then X_F is defined by $dF|_S - \mathrm{id}$, since this latter is a nonzero morphism from $TM|_S$ to $TM|_S$ which vanishes on TS and thus passes to the quotient N_S.

Definition 2.24. We say that F is *tangential* to S if $X_F(N_S^{\otimes \nu_F}) \subseteq TS$.

We define the set $\mathrm{Sing}(F)$ of singularities of F to be the set of points of S where X_F is zero.

For a germ F of diffeomorphism at O in $\mathbb{C}^2$ tangent to the identity, O is dicritical if and only if the blow-up $\tilde{F}$ of F is non-tangential on the exceptional divisor of the blow-up of $\mathbb{C}^2$ at O.

It is worth noticing that being tangential is actually a local condition (if S is connected). That is to say, if $p \notin \mathrm{Sing}(F)$, then F is tangential to S if and only if there exists an open neighborhood U of p such that $X_{F,q}(N_{S,q}^{\otimes \nu_F}) \subseteq T_q S$ for all $q \in U \setminus \{p\}$.

For tangential germs we do have residues theorems:

Theorem 2.25 (Abate, Bracci, Tovena)**.** *Let M be a complex manifold of dimension n, $F : M \to M$ be a holomorphic map having a nonsingular compact hypersurface S as fixed points locus. Assume that F is tangential to S. Let Sing$(F) = \cup_\lambda \Sigma_\lambda$ be the connected components decomposition. Then there exist complex numbers Res$(F, S; \Sigma_\lambda)$ such that*

$$\sum_\lambda \mathrm{Res}(F, S; \Sigma_\lambda) = \int_S c_1^{n-1}(N_S).$$

The residues $\mathrm{Res}(F, S; \Sigma_\lambda)$ are computed in terms of Grothendieck's residues in case Σ_λ is a single point. Theorem 2.25 was proved first by Abate [2] in case $n = 2$, then generalized to the case S is singular in [14]. A proof in terms of foliations in the optic explained before (and for $n = 2$) is in [9]. Finally in [6] the theorem has been proved for any n, for S of any codimension and possibly singular, and also some other indices theorems are provided in case F is non-tangential (but S satisfies some suitable embeddability conditions).

The canonical section reads the dynamics outside singularities. Indeed we have

Theorem 2.26 (Abate, Bracci, Tovena)**.** *Let M be a complex manifold of dimension n, $F : M \to M$ be a holomorphic map having a nonsingular hypersurface S as fixed points locus. Assume that $p \in S$ is such that $p \notin Sing(X_F)$. Then*

(1) *If F is tangential to S then there exists a open neighborhood U of p such that for all $q \in U \setminus S$ there exists $k_0 = k_0(q)$ such that $F^{\circ k}(q) \notin U \setminus S$ for $k > k_0$.*
(2) *If F is non-tangential to S, $X_{F,p}(N_{S,p}^{\otimes \nu_F}) \oplus T_pS = T_pM$ and $\nu_F > 1$ then there exists at least one parabolic curve for F at p tangent to $X_{F,p}(N_{S,p}^{\otimes \nu_F})$.*
(3) *If F is non-tangential to S, $X_{F,p}(N_{S,p}^{\otimes \nu_F}) \oplus T_pS = T_pM$ and $\nu_F = 1$, then there exists "almost always" an F-invariant curve through p on which F is linearizable.*

Notice that the hypothesis $X_{F,p}(N_{S,p}^{\otimes \nu_F}) \oplus T_pS = T_pM$ for F non-tangential to S is a generic condition: if F is non-tangential to S then $X_{F,p}(N_{S,p}^{\otimes \nu_F}) \subseteq T_pS$ only for a discrete set of points.

The "almost always" in part (3) of Theorem 2.26 refers to the action of F on the normal bundle. This action is essentially a number, the only eigenvalue of dF_p not 1 in this case, and the condition is fulfilled if this number has modulus < 1 or > 1, or if it satisfies some Bryuno-like condition, thus "almost always".

Theorem 2.26 can be used to show that the point O is dicritical for a germ F in C^2—but actually in $\mathbb{C}^n$ for any n, providing the natural definition of dicritical point—fixing O and tangent to the identity at O if and only if for all but a finite number of directions there exists at least one parabolic curve for F tangent to such a direction.

Now we have all the ingredients to give the proof of Theorem 2.19.

Sketch of the Proof of Theorem 2.19. . If O is dicritical, blowing $\mathbb{C}^2$ up, the blow-up map $\tilde{F}$ is non-tangential on the exceptional divisor D. A direct calculation shows that the action of F on the normal bundle of D in $\tilde{\mathbb{C}}^2$ is the identity and thus necessarily $\nu_{\tilde{F}} > 1$. Thus Theorem 2.26.(2) provides a Zariski open set of points in D where there exists at least one parabolic curve for $\tilde{F}$. Such curves project down to form parabolic curves for F tangent at almost all directions. Thus we may assume that O (and all further singularities) is not dicritical. By the version of Theorem 2.22 for diffeomorphisms discussed above, after a finite number of blow-ups all the singularities of the blow-up $\tilde{F}$ of F are reduced. Using Theorem 2.25 and combinatorics as in [19] (or some other simplified combinatorics as in [9]) one comes up with a $(\star_1)$-reduced singularity at a nonsingular point of the exceptional divisor. But a $(\star_1)$-reduced singularity on a nonsingular curve of fixed points have (up to some changes of coordinates) a form as in Lemma 2.14, and thus one gets parabolic curves for $\tilde{F}$ and then for F. □

The hidden part in the previous proof is the rôle of the residues. We do not want to enter into details here, however, the residues play the same rôle as in the theory of foliations. In particular, Camacho-Sad or Cano's argument implies that *if F is tangential to a nonsingular curve S of fixed points, $p \in S$ is such that Res$(F, S; p) \notin \mathbb{Q}^+ \cup \{0\}$ then there exists a parabolic curve for F at p.* This argument has been pushed forward by the author [9] and F. degli Innocenti [23] who obtained the following result:

Theorem 2.27 (Bracci, degli Innocenti)**.** *Let F be a germ of holomorphic diffeomorphism of $\mathbb{C}^2$ at O. Assume that the fixed points locus of F at O is a locally irreducible curve S. If F is tangential to S and Res$(F, S; O) \notin \mathbb{Q}^+ \cup \{0\}$ then there exists a parabolic curve for F at O.*

The previous result is due to the author in case S is a cusp, while it was proved by degli Innocenti in full generality. The proof is quite involved for one has to follow the variation of the residue according to the process of desingularization of S.

It should also be remarked that Brochero-Martinez [15] made a very detailed study on dicritical points. In particular he proved

Theorem 2.28 (Brochero-Martinez)**.** *Let F be a germ of holomorphic diffeomorphism of $\mathbb{C}^2$ fixing O, tangent to the identity at O. Assume that O is dicritical for F, let $\tilde{F}$ be the blow-up of F and let D be the exceptional divisor. Then there exist two open sets U^+, U^- in $\tilde{\mathbb{C}}^2$ such that $\overline{U^+ \cup U^-}$ is a neighborhood of $D \setminus$ Sing$(\tilde{F})$ and*

(1) *for all $p \in U^+$ the sequence $F^{\circ k}(p)$ converges to a point of D, as $k \to +\infty$,*
(2) *for all $p \in U^-$ the sequence $F^{\circ -k}(p)$ converges to a point of D, as $k \to +\infty$.*

In particular Theorem 2.28 gives information also on the existence of basins of attraction for F (and F^{-1}) in the dicritical case. Also, in the same paper [15] Brochero-Martinez gives a (semi-)formal classification of dicritical germs.

In dimension greater than two is presently unknown whether all germs tangent to the identity have parabolic curves. Surprisingly enough, a similar construction to the one presented by Gomez-Mont and Luengo [28] for giving an example of holomorphic foliation in $\mathbb{C}^3$ without separatrices, performed in [5] by Abate and Tovena, does not produce the expected counterexample. Indeed, if one calls *robust* the parabolic curves which survive blow-ups, the construction made in [5] produces example of germs tangent to the identity in $\mathbb{C}^3$ with no robust parabolic curves. Nonetheless such examples do have (non-robust) parabolic curves.

We end up this survey by recalling a recent work by Suwa and the author [12] where it is proved the existence of parabolic curves for germs of holomorphic diffeomorphisms tangent to the identity at a singular point of a two dimensional subvariety (under some condition on the type of singularity).

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